# Junction of a periodic family of elastic rods with a 3d plate. Part I.

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**Abstract**

We consider a set of elastic rods periodically distributed over a 3d elastic plate (both of them with axis $x_3$) and we investigate the limit behavior of this problem as the periodicity $\varepsilon$ and the radius $r$ of the rods tend to zero (see fig.1 below). We use a decomposition of the displacement field in the rods of the form $u = U + \overline{u}$ where the principal part $U$ is a field which is piecewise constant with respect to the variables $(x_1, x_2)$ (and then naturally extended on a fixed domain), while the perturbation $\overline{u}$ remains defined on the oscillating domain containing the rods. We derive estimates of $U$ and $\overline{u}$ in term of the total elastic energy. This allows to obtain *a priori* estimates on $u$ without solving the delicate question of the dependence, with respect to $\varepsilon$ and $r$, of the constant in Korn's inequality in such an oscillating domain. To deal with the field $\overline{u}$, we use a version of an unfolding operator which permits both to rescale all the rods and to work on the same fixed domain as for $U$ to carry out the homogenization process. The above decomposition also helps in passing to the limit and to identify the limit junction conditions between the rods and the 3d plate.

**Résumé**

Nous considérons un ensemble de poutres élastiques périodiquement distribuées sur une plaque élastique 3d (toutes d'axe $x_3$) et nous analysons le comportement limite de ce problème lorsque la périodicité $\varepsilon$ et le rayon $r$ des poutres tendent vers zéro. Nous introduisons une décomposition du champ de déplacement de la forme $u = U + \overline{u}$ dans laquelle la partie principale $U$ est un champ constant par morceau par rapport aux variables $(x_1, x_2)$ (et qui s' étend donc naturellement sur un domaine fixe), alors que la perturbation $\overline{u}$ reste un champ défini sur le domaine oscillant qui représente les poutres. Nous donnons des estimations de $U$ et $\overline{u}$ en fonction de l'énergie élastique totale. Ceci permet d'obtenir des estimations *a priori* de $u$ sans chercher à évaluer la dépendance, par rapport à $\varepsilon$ et $r$, de la constante de l'inégalité de Korn pour un tel domaine oscillant. Pour traiter le champ $\overline{u}$, nous utilisons une version d'opérateur d' éclatement qui permet simultanément de redimensionner toutes les poutres et de travailler sur le

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même domaine fixe que pour $U$ afin d'analyser le problème d'homogénéisation. La décomposition ci-dessus facilite aussi le passage à la limite et l'obtention les conditions de jonction limites entre les poutres et la plaque 3d.

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# 1 Introduction

This paper is devoted to describe the asymptotic behavior of an elastic multistructure composed of a set of periodic elastic rods in junction with a 3d plate (see Figure 1). The diameter of each rod tends to zero as the periodicity vanishes, while the height of the rods remains constant. The lateral boundary of the plate is assumed to be clamped. The mechanical model under investigation is the isotropic linearized elasticity system (see e.g. [6]). In this first paper, we consider a plate of constant thickness. The case of the vanishing thickness for the plate is investigated in the second paper [3].

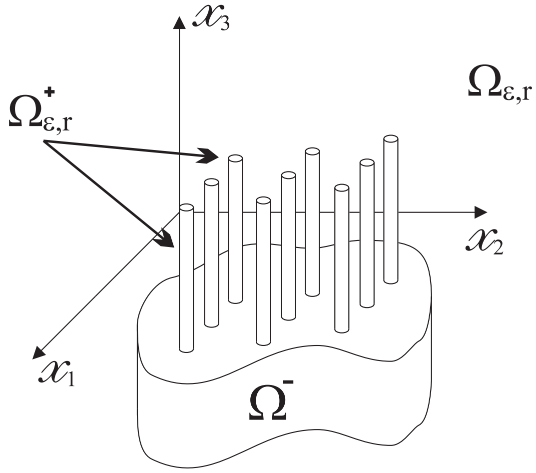


Figure 1: Elastic multistructure with highly oscillating boundary

Since the periodicity and the diameters of the rods tend to zero, while the height of the rods remains constant, this problem pertains to the field of elliptic problems posed on a domain which has a so called: "highly oscillating boundary". Boundary-value problems involving rough boundaries or interfaces appear in many fields of physics and engineering sciences, such as the scattering of acoustic waves on small periodic obstacles, the free vibrations of elastic bodies, the behavior of fluids over rough walls, or of coupled fluid-solid periodic structures. There is a long list of paper concerning domains with highly oscillating boundary (for scalar problems, see e.g. [1], [2], [4], [10], [12], [13] and [22]). Precisely, in [4] the limit problem for the Laplace equation with the homogeneous Neumann boundary condition and with a $L^2$-right-hand side is derived. For the same problem, a nonoscillating approximation of the solution at order $\mathcal{O}\left(\varepsilon^{1-\delta}\right)$ in the $H^1$-norm is obtained in [22], under an additional assumption on the right-hand side. In the case of the Laplace equation with Dirichlet boundary conditions, a nonoscillating approximation of the solution at order $\mathcal{O}\left(\varepsilon^{\frac{3}{2}}\right)$ in the $H^1$-norm is constructed in [1]. The Laplace equation with a non-homogeneous Neumann boundary condition is studied in [13]. The limit energy of the p-laplacian is obtained in [10], while a corresponding monotone problem is considered in [2]. The optimal control for a parabolic problem is studied in [12]. For the asymptotic behaviour of transmission problems, we refer to [14] and [18]. For general references about domains with singular perturbations and multidomain, we refer to [9], [19], [20], [21], [25]. For mathematical modelling of rods we refer to [23], [24] and [27]. For a presentation of the homogenization theory we refer to [26].

Even if our model is linear isotropic elasticity, the vectorial character of the unknown (the 3d displacement) precludes from reproducing the analysis used for the above scalar problems to take into account the fast oscillations of the rods. Indeed, the first difference concerns the derivation of a priori estimates on the displacement (or the stress) field: the dependance of the constant in Korn's inequality with respect to the period $\varepsilon$ of the rods and their diameter $r$ is not relevant. In some sense this is due to very different behavior of the displacements in the rods and in the plate. To overcome this first difficulty we use a decomposition of the $3d$ displacement in the rods introduced in [16] and [17], which involves the mean displacement and the main rotation of each cross section of each rod (see Section 3). The main property of this decomposition relies on a priori estimates of its terms with bounds depending on $\varepsilon$, $r$ and the total elastic energy. Loosely speaking, this leads to estimates of the type:

$$\|u_i^{\varepsilon,r}\|^2_{L^2(\Omega^+_{\varepsilon,r})} \leq c_i(\varepsilon,r)\mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r}), \quad i=1,2,3,$$

where $u^{\varepsilon,r}$ is the displacement in the set of rods $\Omega^+_{\varepsilon,r}$, $c_i(\varepsilon,r)$ is a constant which depends on $\varepsilon$, $r$ and on the component of the displacement, and $\mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r})$ is the total elastic energy in the rods $\Omega^+_{\varepsilon,r}$ and in the plate $\Omega^-$: that is $\Omega_{\varepsilon,r}=\Omega^+_{\varepsilon,r}\cup\Omega^-$. This process allows to precise the scaling of the applied forces and to obtain more precise estimates on the displacement (or on its decomposition) than by using Korn's inequality. The second difficulty arises when passing to limit as $\varepsilon$ and $r$ tend to 0; indeed the solution is defined on a domain $\Omega_{\varepsilon,r}$ which depends on $\varepsilon$ and $r$. In the scalar case, it is sufficient to extend the solution by 0 outside $\Omega^+_{\varepsilon,r}$ and to remark that the derivative in the direction of the axis of the rods (say $x_3$) commutes with this extension process. It is well known that this simple argument does not work in elasticity in order to describe the bending in the rods (the only deformation which commutes with the 0-extension is $\partial_{x_3}u_3$). Actually, the decomposition we use for the displacement also helps passing to the limit: it provides an approximation of the $3d$ displacement in the rods which is defined on a fixed domain (the domain asymptotically filled by the rods). Indeed, the mean displacement and the mean rotation of each rod lead to functions of $x_3$ which are piecewise constant with respect to $(x_1,x_2)$. To deal with the rest of the decomposition, i.e. the part which remains a field of $(x_1,x_2,x_3)$, we use first the a priori estimates (in terms of the elastic energy) mentioned above and then a tool developed in [8], referred as the unfolding operator technique, which also allows to work on a fixed domain (but with more variables). A similar technique has been used in [5] for reticulated elastic structure. Let us emphasize that with such an approach we not only identify the limit problem as a "continuum" model of $1d$ rods coupled with $3d$ elasticity in the plate; but we also show that the relevant physical quantities (the mean of the $3d$ displacement in the cross-section of each rod) converge (in adapted norms) to the solution of the limit problem. References and other applications of the unfolding operator technique can be found in [7], [11] and [15].

The paper is organizes as follows. In Section 2 we describe the geometry and the model under consideration and specify the assumptions on the applied forces. Section 3 is devoted to introduce the decomposition of the displacement field $u^\varepsilon$ in the rods. In Section 4, we derive the a priori estimates on each rods. In Section 5 we introduce the unfolding operator and derive the estimates on the unfold fields. We also obtain the junction conditions between the limit model for the rods and the plate. We first pass to the limit in Section 6 in the case where the radius of the rods $r$ is of order $\varepsilon$. In Section 7 we examine the case $r = o(\varepsilon)$. At least in Section 8 we prove convergence of the energies and deduce a few strong convergence results of the fields. Section 9 is devoted to summarize the results.

# 2 Position of the problem

We investigate the behavior of an elastic $3d$ body $\Omega_{\varepsilon,r}$ composed of two parts: a forest of rods $\Omega^+_{\varepsilon,r}$ and a $3d$ plate $\Omega^-$.

To describe the geometry of $\Omega^+_{\varepsilon,r}$, let us consider an open bounded domain $\omega$ with Lipschitz boundary contained in the $(x_1, x_2)$-coordinate plane. For a real number $\varepsilon > 0$, $\mathcal{N}_\varepsilon$ denotes the following subset of $\mathbb{Z}^2$:

$$\mathcal{N}_\varepsilon = \Big\{(p, q) \in \mathbb{Z}^2 : \Big]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\Big[ \times \Big]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\Big[ \subset \omega\Big\}. \tag{2.1}$$

Fix $L > 0$. For each $(p, q) \in \mathbb{Z}^2$, $\varepsilon > 0$ and $r > 0$, we consider a rod $\mathcal{P}^{\varepsilon,r}_{pq}$ whose cross section is the disk of center $(\varepsilon p, \varepsilon q)$ and radius $r$, and whose axis is $x_3$ and which has a height equal to $L$:

$$\mathcal{D}^{\varepsilon,r}_{pq} = \big\{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - \varepsilon p)^2 + (x_2 - \varepsilon q)^2 < r^2\big\}, \tag{2.2}$$

$$\mathcal{P}^{\varepsilon,r}_{pq} = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \mathcal{D}^{\varepsilon,r}_{pq},\ 0 < x_3 < L\big\}. \tag{2.3}$$

Then, for $r \in \Big]0, \dfrac{\varepsilon}{2}\Big[$, we denote by $\Omega^+_{\varepsilon,r}$ the set of all the rods defined as above:

$$\Omega^+_{\varepsilon,r} = \bigcup_{(p,q)\in\mathcal{N}_\varepsilon} \mathcal{P}^{\varepsilon,r}_{pq}. \tag{2.4}$$

The lower cross sections of all the rods is denoted by $\omega_{\varepsilon,r}$:

$$\omega_{\varepsilon,r} = \bigcup_{(p,q)\in\mathcal{N}_\varepsilon} \mathcal{D}^{\varepsilon,r}_{pq} \times \{0\} \subset \omega. \tag{2.5}$$

We have assumed that $r \le \dfrac{\varepsilon}{2}$, in order to avoid the contact between two different rods.

The $3d$ plate is defined by

$$\Omega^- = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \omega,\ -l < x_3 < 0\big\}, \tag{2.6}$$

where $l$ is a positive fixed real number.

The elastic body $\Omega_{\varepsilon,r}$ is defined by

$$\Omega_{\varepsilon,r} = \Omega^+_{\varepsilon,r} \cup \omega_{\varepsilon,r} \cup \Omega^-. \tag{2.7}$$

The domain asymptotically filled by the oscillating part $\Omega^+_{\varepsilon,r}$ of $\Omega_{\varepsilon,r}$ (as $\varepsilon$ tends to zero) is denoted by $\Omega^+$:

$$\Omega^+ = \omega \times ]0, L[. \tag{2.8}$$

Moreover, $\Omega$ is defined by

$$\Omega = \omega \times ] - l, L[. \tag{2.9}$$

We consider the standard linear isotropic equations of elasticity in $\Omega_{\varepsilon,r}$.

The displacement field in $\Omega_{\varepsilon,r}$ is denoted by

$$u^{\varepsilon,r} : \Omega_{\varepsilon,r} \longrightarrow \mathbb{R}^3.$$

The linearized deformation field in $\Omega_{\varepsilon,r}$ is defined by

$$\gamma(u^{\varepsilon,r}) = \frac{1}{2}\left(Du^{\varepsilon,r} + (Du^{\varepsilon,r})^T\right), \tag{2.10}$$

or equivalently by its components:

$$\gamma_{ij}(u^{\varepsilon,r}) = \frac{1}{2}\left(\partial_i u^{\varepsilon,r}_j + \partial_j u^{\varepsilon,r}_i\right), \quad i, j = 1, 2, 3. \tag{2.11}$$

The Cauchy stress tensor in $\Omega_{\varepsilon,r}$ is linked to $\gamma(u^{\varepsilon,r})$ through the standard Hooke's law:

$$\sigma^{\varepsilon,r} = \lambda \left(\mathrm{Tr}\, \gamma(u^{\varepsilon,r})\right) I + 2\mu\gamma(u^{\varepsilon,r}), \tag{2.12}$$

where $\lambda$ and $\mu$ denotes the Lamé coefficients of the elastic material, and $I$ is the identity $3 \times 3$ matrix. Indeed (2.12) writes as

$$\sigma^{\varepsilon,r}_{ij} = \lambda \left(\sum_{k=1}^{3} \gamma_{kk}(u^{\varepsilon,r})\right) \delta_{ij} + 2\mu\gamma_{ij}(u^{\varepsilon,r}), \quad i, j = 1, 2, 3, \tag{2.13}$$

where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.

The equation of equilibrium in $\Omega_{\varepsilon,r}$ writes as

$$-\sum_{j=1}^{3} \partial_j \sigma^{\varepsilon,r}_{ij} = f^{\varepsilon,r}_i \text{ in } \Omega_{\varepsilon,r}, \quad i = 1, 2, 3, \tag{2.14}$$

where $f^{\varepsilon,r} : \Omega_{\varepsilon,r} \longrightarrow \mathbb{R}^3$ denotes the volume applied force.

In order to specify the boundary conditions on $\partial\Omega_{\varepsilon,r}$, we will assume that:

- the $3d$ plate is clamped on its lateral boundary $\partial\omega \times ] - l, 0[= \Gamma_{\text{lat}}$:

$$u^{\varepsilon,r} = 0 \text{ on } \Gamma_{\text{lat}}, \tag{2.15}$$

- the boundary $\partial\Omega_{\varepsilon,r} \setminus \Gamma_{\text{lat}}$ is free:

$$\sigma^{\varepsilon,r}\nu = 0 \text{ on } \partial\Omega_{\varepsilon,r} \setminus \Gamma_{\text{lat}}, \tag{2.16}$$

where $\nu$ denotes the exterior unit normal to $\Omega_{\varepsilon,r}$.

**Remark 2.1.** *Assumption (2.16) means that the density of applied surface forces on the boundary $\partial\Omega_{\varepsilon,r} \setminus \Gamma_{lat}$ is zero. This assumption is not necessary to carry on the analysis, but it is a bit natural as far as the fast oscillating boundary $\partial\Omega^+_{\varepsilon,r}$ is concerned.*

The variational formulation of (2.14)÷(2.16) is very standard. If $V_{\varepsilon,r}$ denotes the space:

$$V_{\varepsilon,r} = \left\{ v \in \left(H^1(\Omega_{\varepsilon,r})\right)^3 : v = 0 \text{ on } \Gamma_{\text{lat}} \right\}, \tag{2.17}$$

it results that

$$\begin{cases} u^{\varepsilon,r} \in V_{\varepsilon,r}, \\ \displaystyle\int_{\Omega_{\varepsilon,r}} \sum_{i,j=1}^{3} \sigma_{ij}^{\varepsilon,r} \gamma_{ij}(v)dx = \int_{\Omega_{\varepsilon,r}} \sum_{i=1}^{3} f_i^{\varepsilon,r} v_i dx, \quad \forall v \in V_{\varepsilon,r}. \end{cases} \tag{2.18}$$

As far as the assumption on the applied forces is concerned, we assume that throughout the paper

$$f_\alpha^{\varepsilon,r} = r f_\alpha \text{ in } \Omega^+_{\varepsilon,r}, \text{ for } \alpha = 1, 2, \tag{2.19}$$

$$f_3^{\varepsilon,r} = f_3 \text{ in } \Omega^+_{\varepsilon,r}, \tag{2.20}$$

$$f_i^{\varepsilon,r} = f_i \text{ in } \Omega^-, \text{ for } i = 1, 2, 3, \tag{2.21}$$

where $f \in \left(L^2(\Omega)\right)^3$ is given.

# 3 Decomposition of the displacement in $\Omega^+_{\varepsilon,r}$ and estimates in $\Omega^-$

As usual, to obtain *a priori* estimates on $u^{\varepsilon,r}$, then on $\gamma(u^{\varepsilon,r})$ and $\sigma^{\varepsilon,r}$, we plug the test function $u^{\varepsilon,r}$ in (2.18) to obtain

$$\int_{\Omega_{\varepsilon,r}} \sum_{i,j=1}^{3} \sigma_{ij}^{\varepsilon,r} \gamma_{ij}(u^{\varepsilon,r})dx = \int_{\Omega_{\varepsilon,r}} \sum_{i=1}^{3} f_i^{\varepsilon,r} u_i^{\varepsilon,r} dx. \tag{3.1}$$

The main difficulty in deriving *a priori* estimates from (3.1) is the dependance upon $r$ and $\varepsilon$ in the Korn's inequality in $\Omega_{\varepsilon,r}$. Indeed, this is due to the fast oscillating part $\Omega^+_{\varepsilon,r}$ (in $\Omega^-$ Korn's inequality is standard and the boundary condition (2.15) permits to control $\|u_i^{\varepsilon,r}\|_{L^2(\Omega^-)}$). Moreover, for a multi-structure like $\Omega_{\varepsilon,r}$, it is not very convenient to estimate the constant in a Korn's type inequality because the order of each component of the displacement field (say in $L^2$-norm, with respect to $\varepsilon$ and $r$) may be very different. To overcome this difficulty, in the sequel we will use a decomposition of the field $u^{\varepsilon,r}$ in each rod $\mathcal{P}^{\varepsilon,r}_{pq}$, which, in some sense, takes advantage of the geometry of a rod (see [17]).

Fix $\varepsilon$, $r$, and $(p,q)$ in $\mathcal{N}_\varepsilon$ and let us drop the index $\varepsilon$, $r$ and $(p,q)$ in $\mathcal{D}^{\varepsilon,r}_{pq}$ and $\mathcal{P}^{\varepsilon,r}_{pq}$ (then for a while, $\mathcal{D}$ and $\mathcal{P}$ denote $\mathcal{D}^{\varepsilon,r}_{pq}$ and $\mathcal{P}^{\varepsilon,r}_{pq}$).

For any displacement $v \in (H^1(\mathcal{O}))^3$ of a open smooth domain $\mathcal{O}$, the elastic energy is denoted by

$$\mathcal{E}_{\mathcal{O}}(v) = \int_{\mathcal{O}} \left[ \lambda \left( \sum_{k=1}^{3} \gamma_{kk}(v) \right)^2 + 2\mu \sum_{i,j=1}^{3} (\gamma_{ij}(v))^2 \right] dx. \tag{3.2}$$

In order to obtain a useful decomposition of $v$, we introduce the following notations:

$$\mathcal{U}(x_3) = \frac{1}{\pi r^2} \int_{\mathcal{D}} v(x_1,x_2,x_3) dx_1 dx_2, \tag{3.3}$$

$$\mathcal{R}_1(x_3) = \frac{1}{I_2 r^4} \int_{\mathcal{D}} (x_2 - \varepsilon q) v_3(x_1,x_2,x_3) dx_1 dx_2, \tag{3.4}$$

$$\mathcal{R}_2(x_3) = -\frac{1}{I_1 r^4} \int_{\mathcal{D}} (x_1 - \varepsilon p) v_3(x_1,x_2,x_3) dx_1 dx_2, \tag{3.5}$$

$$\mathcal{R}_3(x_3) = \frac{1}{(I_1+I_2) r^4} \int_{\mathcal{D}} (x_1 - \varepsilon p) v_2(x_1,x_2,x_3) - (x_2 - \varepsilon q) v_1(x_1,x_2,x_3) dx_1 dx_2, \tag{3.6}$$

where $I_1 = \dfrac{1}{r^4} \displaystyle\int_{\mathcal{D}} (x_1 - \varepsilon p)^2 dx_1 dx_2 = \dfrac{\pi}{4} = \dfrac{1}{r^4} \displaystyle\int_{\mathcal{D}} (x_2 - \varepsilon q)^2 dx_1 dx_2 = I_2$.

Let us denote by $\mathcal{R}$ the vectorial field $(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3)$ and set

$$\overline{v}(x_1,x_2,x_3) = v(x_1,x_2,x_3) - \mathcal{U}(x_3) - \mathcal{R}(x_3) \wedge ((x_1 - \varepsilon p) e_1 + (x_2 - \varepsilon q) e_2). \tag{3.7}$$

where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$.

Indeed, due to the definition of $\mathcal{R}$ and to the symmetry of $\mathcal{D}$, one has that

$$\int_{\mathcal{D}} \overline{v}_i(x_1,x_2,x_3) dx_1 dx_2 = 0, \qquad \text{for } i = 1,2,3, \tag{3.8}$$

$$\int_{\mathcal{D}} (x_1 - \varepsilon p) \overline{v}_3(x_1,x_2,x_3) dx_1 dx_2 = \int_{\mathcal{D}} (x_2 - \varepsilon q) \overline{v}_3(x_1,x_2,x_3) dx_1 dx_2 = 0, \tag{3.9}$$

$$\int_{\mathcal{D}} (x_1 - \varepsilon p) \overline{v}_2(x_1,x_2,x_3) - (x_2 - \varepsilon q) \overline{v}_1(x_1,x_2,x_3) dx_1 dx_2 = 0, \tag{3.10}$$

for almost any $x_3$ in $]0,L[$.

The following lemma is proved in [16].

**Lemma 3.1.** *For $L > r$, there exists a constant $c$ (which does not depend on $L$ and $r$) such that for any $v \in (H^1(\mathcal{P}))^3$:*

$$\left\| \frac{d\mathcal{U}}{dx_3} - \mathcal{R} \wedge e_3 \right\|^2_{(L^2]0,L[)^3} \leq \frac{c}{r^2} \mathcal{E}_{\mathcal{P}}(v), \tag{3.11}$$

$$\left\| \frac{d\mathcal{R}}{dx_3} \right\|^2_{(L^2]0,L[)^3} \le \frac{c}{r^4}\mathcal{E}_{\mathcal{P}}(v), \tag{3.12}$$

$$\|\overline{v}\|^2_{(L^2(\mathcal{P}))^3} \le cr^2\mathcal{E}_{\mathcal{P}}(v), \tag{3.13}$$

$$\|D\overline{v}\|^2_{(L^2(\mathcal{P}))^9} \le c\mathcal{E}_{\mathcal{P}}(v), \tag{3.14}$$

*where* $\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3)$, $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3)$ *and* $\overline{v}$ *are defined in (3.3)÷(3.7).*

To end this section, we recall that, since $u^{\varepsilon,r} = 0$ on $\partial\omega\times]-l,0[$, Korn's inequality yields:

$$\|u^{\varepsilon,r}\|^2_{(L^2(\Omega^-))^3} + \|Du^{\varepsilon,r}\|^2_{(L^2(\Omega^-))^9} \le c\mathcal{E}_{\Omega^-}(u^{\varepsilon,r}) = c\int_{\Omega^-} \sum_{i,j=1}^{3} \sigma_{ij}^{\varepsilon,r}\gamma_{ij}(u^{\varepsilon,r})dx, \tag{3.15}$$

where $c$ is a constant independent of $\varepsilon$ and $r$.

# 4 *A priori* estimates

Let us consider the displacement $u^{\varepsilon,r} \in (H^1(\Omega_{\varepsilon,r}))^3$ solution of (2.14)÷(2.16). Indeed, $u^{\varepsilon,r} \in \left(H^1(\mathcal{P}_{pq}^{\varepsilon,r})\right)^3$, for any $(p,q) \in \mathcal{N}^\varepsilon$. Then, the previous section permits to define, for any $(p,q) \in \mathcal{N}^\varepsilon$, the fields $\mathcal{U}_{pq}^{\varepsilon,r}$, $\mathcal{R}_{pq}^{\varepsilon,r}$ and $\overline{u}_{pq}^{\varepsilon,r}$, through the formulae (3.3)÷(3.7), with $u^{\varepsilon,r}$ in place of $v$. Recall that for any $(p,q) \in \mathcal{N}^\varepsilon$, $\mathcal{U}_{pq}^{\varepsilon,r} \in (H^1(]0,L[))^3$, $\mathcal{R}_{pq}^{\varepsilon,r} \in (H^1(]0,L[))^3$, and $\overline{u}_{pq}^{\varepsilon,r} \in \left(H^1(\mathcal{P}_{pq}^{\varepsilon,r})\right)^3$.

In order to shorten the notation, we set:

$$\widetilde{\omega}_\varepsilon = \bigcup_{(p,q)\in\mathcal{N}^\varepsilon} \left( \left]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\right[ \times \left]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\right[ \right) \subset \omega. \tag{4.1}$$

Now we define the field $\mathcal{U}^{\varepsilon,r}$ and $\mathcal{R}^{\varepsilon,r}$ almost everywhere in $\Omega^+$ by

$$\mathcal{U}^{\varepsilon,r}(x_1,x_2,x_3) = \mathcal{U}_{pq}^{\varepsilon,r}(x_3), \text{ if } (x_1,x_2) \in \left]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\right[ \times \left]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\right[, \tag{4.2}$$

$$\mathcal{R}^{\varepsilon,r}(x_1,x_2,x_3) = \mathcal{R}_{pq}^{\varepsilon,r}(x_3), \text{ if } (x_1,x_2) \in \left]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\right[ \times \left]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\right[, \tag{4.3}$$

$$\mathcal{U}^{\varepsilon,r}(x_1,x_2,x_3) = \mathcal{R}^{\varepsilon,r}(x_1,x_2,x_3) = 0, \quad \text{if } (x_1,x_2) \in \omega \setminus \widetilde{\omega}_\varepsilon, \tag{4.4}$$

which means that $\mathcal{U}^{\varepsilon,r}(\cdot,\cdot,x_3)$ and $\mathcal{R}^{\varepsilon,r}(\cdot,\cdot,x_3)$ are constants on each cell $\left]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\right[ \times \left]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\right[$.

Indeed, we have that $\mathcal{U}^{\varepsilon,r}$, $\mathcal{R}^{\varepsilon,r} \in (L^2(\Omega^+))^3$, and for $i = 1,2,3$

$$\|\mathcal{U}_i^{\varepsilon,r}\|^2_{L^2(\Omega^+)} = \varepsilon^2 \sum_{(p,q)\in\mathcal{N}^\varepsilon} \int_0^L \left| \left(\mathcal{U}_{pq}^{\varepsilon,r}\right)_i(x_3)\right|^2 dx_3 = \varepsilon^2 \sum_{(p,q)\in\mathcal{N}^\varepsilon} \| \left(\mathcal{U}_{pq}^{\varepsilon,r}\right)_i \|^2_{L^2(]0,L[)}, \tag{4.5}$$

$$\|\mathcal{R}_i^{\varepsilon,r}\|^2_{L^2(\Omega^+)} = \varepsilon^2 \sum_{(p,q)\in\mathcal{N}^\varepsilon} \int_0^L \left| \left(\mathcal{R}_{pq}^{\varepsilon,r}\right)_i(x_3)\right|^2 dx_3 = \varepsilon^2 \sum_{(p,q)\in\mathcal{N}^\varepsilon} \| \left(\mathcal{R}_{pq}^{\varepsilon,r}\right)_i \|^2_{L^2(]0,L[)}. \tag{4.6}$$

Moreover, since

$$\frac{\partial \mathcal{U}^{\varepsilon,r}}{\partial x_3}(x_1,x_2,x_3)=\frac{d\mathcal{U}^{\varepsilon,r}_{pq}}{dx_3}(x_3),\ \text{if } (x_1,x_2)\in\Big]\varepsilon p-\frac{\varepsilon}{2},\varepsilon p+\frac{\varepsilon}{2}\Big[\times\Big]\varepsilon q-\frac{\varepsilon}{2},\varepsilon q+\frac{\varepsilon}{2}\Big[,\tag{4.7}$$

and

$$\frac{\partial \mathcal{R}^{\varepsilon,r}}{\partial x_3}(x_1,x_2,x_3)=\frac{d\mathcal{R}^{\varepsilon,r}_{pq}}{dx_3}(x_3),\ \text{if } (x_1,x_2)\in\Big]\varepsilon p-\frac{\varepsilon}{2},\varepsilon p+\frac{\varepsilon}{2}\Big[\times\Big]\varepsilon q-\frac{\varepsilon}{2},\varepsilon q+\frac{\varepsilon}{2}\Big[,\tag{4.8}$$

it follows that

$$\mathcal{U}^{\varepsilon,r},\mathcal{R}^{\varepsilon,r}\in\left(L^2\left(\omega,H^1(]0,L[)\right)\right)^3,\tag{4.9}$$

(recall that $\mathcal{U}^{\varepsilon,r}_{pq},\mathcal{R}^{\varepsilon,r}_{pq}\in(H^1(]0,L[))^3$, for any $(p,q)\in\mathcal{N}^\varepsilon$) and for $i=1,2,3$

$$\left\|\frac{\partial \mathcal{U}^{\varepsilon,r}_i}{\partial x_3}\right\|^2_{L^2(\Omega^+)}=\varepsilon^2\sum_{(p,q)\in\mathcal{N}^\varepsilon}\int_0^L\left|\left(\frac{d\mathcal{U}^{\varepsilon,r}_{pq}}{dx_3}\right)_i\right|^2dx_3=\varepsilon^2\sum_{(p,q)\in\mathcal{N}^\varepsilon}\left\|\left(\frac{d\mathcal{U}^{\varepsilon,r}_{pq}}{dx_3}\right)_i\right\|^2_{L^2(]0,L[)},\tag{4.10}$$

$$\left\|\frac{\partial \mathcal{R}^{\varepsilon,r}_i}{\partial x_3}\right\|^2_{L^2(\Omega^+)}=\varepsilon^2\sum_{(p,q)\in\mathcal{N}^\varepsilon}\int_0^L\left|\left(\frac{d\mathcal{R}^{\varepsilon,r}_{pq}}{dx_3}\right)_i\right|^2dx_3=\varepsilon^2\sum_{(p,q)\in\mathcal{N}^\varepsilon}\left\|\left(\frac{d\mathcal{R}^{\varepsilon,r}_{pq}}{dx_3}\right)_i\right\|^2_{L^2(]0,L[)}.\tag{4.11}$$

As far as the set of functions $\overline{u}^{\varepsilon,r}_{pq}$ are concerned, we define the function $\overline{u}^{\varepsilon,r}$ a.e. in $\Omega^+_{\varepsilon,r}$ by

$$\overline{u}^{\varepsilon,r}=\overline{u}^{\varepsilon,r}_{pq},\ \text{if } (x_1,x_2,x_3)\in\mathcal{P}^{\varepsilon,r}_{pq}.\tag{4.12}$$

In order to obtain estimates on the quantities $\mathcal{U}^{\varepsilon,r}$, $\mathcal{R}^{\varepsilon,r}$, $\overline{u}^{\varepsilon,r}$ and $u^{\varepsilon,r}$ in various norm, the strategy is the following. At first, we derive a few estimates on the fields $\mathcal{U}^{\varepsilon,r}$, $\mathcal{R}^{\varepsilon,r}$, $\overline{u}^{\varepsilon,r}$ and $u^{\varepsilon,r}$ respectively in terms of the total elastic energy:

$$\mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r})=\int_{\Omega_{\varepsilon,r}}\sum_{i,j=1}^3\sigma^{\varepsilon,r}_{ij}\gamma_{ij}(u^{\varepsilon,r})dx.$$

Then, we use (3.1) and assumptions (2.19), (2.20), (2.21) on the forces $(f^{\varepsilon,r}_i)$ to obtain an uniform estimates on $\mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r})$, from which we deduce uniform bounds on $\mathcal{U}^{\varepsilon,r}$, $\mathcal{R}^{\varepsilon,r}$, $\overline{u}^{\varepsilon,r}$ and $u^{\varepsilon,r}$.

In the sequel of this Section, $c$ denotes any positive constant independent of $\varepsilon$ and $r$.

## 4.1 Uniform bound on $\mathcal{U}^{\varepsilon,r}$ and $\mathcal{R}^{\varepsilon,r}$ in terms of $\mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r})$

The estimates on $\mathcal{U}^{\varepsilon,r}$ and $\mathcal{R}^{\varepsilon,r}$ are obtained in two steps. In the first step, estimates on $\mathcal{U}^{\varepsilon,r}(\cdot,\cdot,0)$ and $\mathcal{R}^{\varepsilon,r}(\cdot,\cdot,0)$ are derived in term of $\mathcal{E}_{\Omega^-}(u^{\varepsilon,r})$, by using the definitions (3.3)÷(3.6) and estimate (3.15). Then, in step 2, we use (4.7) and (4.8) and estimates (3.11), (3.14) in each road $\mathcal{P}^{\varepsilon,r}_{pq}$.

Step 1. Estimates on $\mathcal{U}^{\varepsilon,r}(\cdot,\cdot,0)$ and $\mathcal{R}^{\varepsilon,r}(\cdot,\cdot,0)$.
We begin with $\mathcal{R}^{\varepsilon,r}(\cdot,\cdot,0)$ and we only detail the technique for $\mathcal{R}^{\varepsilon,r}_1$.

First recall that for any $(p,q) \in \mathcal{N}^\varepsilon$, we have that

$$\left(\mathcal{R}^{\varepsilon,r}_{pq}\right)_1(0) = \frac{1}{I_2 r^4} \int_{\mathcal{D}^{\varepsilon,r}_{pq}} (x_2 - \varepsilon q) u^{\varepsilon,r}_3(x_1,x_2,0) dx_1 dx_2. \tag{4.13}$$

Now $u^{\varepsilon,r}_3(x_1,x_2,0)$ is indeed also the trace on $\mathcal{D}^{\varepsilon,r}_{pq}$ of the displacement $u^{\varepsilon,r}_3$ in $\Omega^-$. Then, by using estimate (3.15), we have

$$\|u^{\varepsilon,r}_3(x_1,x_2,0)\|^2_{L^2(\omega)} \leq c\mathcal{E}_{\Omega^-}(u^{\varepsilon,r}).$$

Consequently, by using the Cauchy-Schwarz's inequality in (4.13) and by summing up all the obtained inequalities over $(p,q) \in \mathcal{N}^\varepsilon$, we get

$$\sum_{(p,q)\in\mathcal{N}^\varepsilon} \left| \left(\mathcal{R}^{\varepsilon,r}_{pq}\right)_1(0) \right|^2 \leq \frac{c}{r^4} \mathcal{E}_{\Omega^-}(u^{\varepsilon,r}). \tag{4.14}$$

Actually we derive a sharper estimate using Poincaré-Wirtinger inequality's and the term $(x_2 - \varepsilon q)$ in definition (4.13) (this will be useful to obtain the junction condition on $\omega$ in the limit problem).

For any $(p,q) \in \mathcal{N}^\varepsilon$, we extend $\left(\mathcal{R}^{\varepsilon,r}_{pq}\right)_1$ for almost $x_3 \in ]-l,0[$ by

$$\left(\mathcal{R}^{\varepsilon,r}_{pq}\right)_1(x_3) = \frac{1}{I_2 r^4} \int_{\mathcal{D}^{\varepsilon,r}_{pq}} (x_2 - \varepsilon q) u^{\varepsilon,r}_3(x_1,x_2,x_3) dx_1 dx_2. \tag{4.15}$$

Indeed $\left(\mathcal{R}^{\varepsilon,r}_{pq}\right)_1 \in H^1(]-l,0[)$, and

$$\frac{d\left(\mathcal{R}^{\varepsilon,r}_{pq}\right)_1}{dx_3}(x_3) = \frac{1}{I_2 r^4} \int_{\mathcal{D}^{\varepsilon,r}_{pq}} (x_2 - \varepsilon q) \frac{\partial u^{\varepsilon,r}_3}{\partial x_3}(x_1,x_2,x_3) dx_1 dx_2. \tag{4.16}$$

If we denote by $\mathcal{M}_{\mathcal{D}^{\varepsilon,r}_{pq}}(u^{\varepsilon,r}_3)(x_3)$ the mean of $u^{\varepsilon,r}_3$ over $\mathcal{D}^{\varepsilon,r}_{pq}$, that is

$$\mathcal{M}_{\mathcal{D}^{\varepsilon,r}_{pq}}(u^{\varepsilon,r}_3)(x_3) = \frac{1}{|\mathcal{D}^{\varepsilon,r}_{pq}|} \int_{\mathcal{D}^{\varepsilon,r}_{pq}} u^{\varepsilon,r}_3(x_1,x_2,x_3) dx_1 dx_2,$$

we first have that

$$\left(\mathcal{R}^{\varepsilon,r}_{pq}\right)_1(x_3) = \frac{1}{I_2 r^4_\varepsilon} \int_{\mathcal{D}^{\varepsilon,r}_{pq}} (x_2 - \varepsilon q) \left[ u^{\varepsilon,r}_3(x_1,x_2,x_3) - \mathcal{M}_{\mathcal{D}^{\varepsilon,r}_{pq}}(u^{\varepsilon,r}_3)(x_3) \right] dx_1 dx_2, \tag{4.17}$$

(and here the term $(x_2 - \varepsilon q)$ plays the important role in the estimate) and secondly, because of Poincaré-Wirtinger inequality's on $\mathcal{D}^{\varepsilon,r}_{pq}$ (which has radius equal to $r$), we have that

$$\left\| u^{\varepsilon,r}_3 - \mathcal{M}_{\mathcal{D}^{\varepsilon,r}_{pq}}(u^{\varepsilon,r}_3) \right\|^2_{L^2(\mathcal{D}^{\varepsilon,r}_{pq}\times]-l,0[)} \leq cr^2 \left\| D_{x_1,x_2} u^{\varepsilon,r}_3 \right\|^2_{\left(L^2(\mathcal{D}^{\varepsilon,r}_{pq}\times]-l,0[)\right)^2}, \tag{4.18}$$

where $D_{x_1,x_2} u^{\varepsilon,r}_3$ denotes the gradient of $u^{\varepsilon,r}_3$ with respect to the variables $x_1, x_2$.

From (4.17) and (4.18), we deduce that, for any $(p,q) \in \mathcal{N}^\varepsilon$,

$$\left\| \left(\mathcal{R}^{\varepsilon,r}_{pq}\right)_1 \right\|^2_{L^2(]-l,0[)} \leq \frac{c}{r^2} \left\| D_{x_1,x_2} u^{\varepsilon,r}_3 \right\|^2_{\left(L^2(\mathcal{D}^{\varepsilon,r}_{pq}\times]-l,0[)\right)^2}. \tag{4.19}$$

Due to (4.16) we have

$$\left\| \frac{d\left(\mathcal{R}^{\varepsilon,r}_{pq}\right)_1}{dx_3} \right\|^2_{L^2(]-l,0[)} \leq \frac{c}{r^4} \left\| \frac{\partial u^{\varepsilon,r}_3}{\partial x_3} \right\|^2_{L^2(\mathcal{D}^{\varepsilon,r}_{pq}\times]-l,0[)} . \tag{4.20}$$

As a consequence of (4.19) and (4.20) it results that

$$\left| \left(\mathcal{R}^{\varepsilon,r}_{pq}\right)_1(0) \right|^2 \leq \frac{c}{r^3} \left\| Du^{\varepsilon,r}_3 \right\|^2_{\left(L^2(\mathcal{D}^{\varepsilon,r}_{pq}\times]-l,0[)\right)^3} . \tag{4.21}$$

By summing up over all $(p,q)\in\mathcal{N}^\varepsilon$, we obtain

$$\sum_{(p,q)\in\mathcal{N}^\varepsilon} \left| \left(\mathcal{R}^{\varepsilon,r}_{pq}\right)_1(0) \right|^2 \leq \frac{c}{r^3} \left\| u^{\varepsilon,r}_3 \right\|^2_{H^1(\Omega^-)}. \tag{4.22}$$

and, with the help of the Korn's inequality in $\Omega^-$ (see (3.15)), we have

$$\sum_{(p,q)\in\mathcal{N}^\varepsilon} \left| \left(\mathcal{R}^{\varepsilon,r}_{pq}\right)_1(0) \right|^2 \leq \frac{c}{r^3} \mathcal{E}_{\Omega^-}(u^{\varepsilon,r}), \tag{4.23}$$

which is an improvement of (4.14).

Now, in view of the definition (4.3)-(4.4) of $\mathcal{R}^{\varepsilon,r}$, we deduce that

$$\left\| \left(\mathcal{R}^{\varepsilon,r}\right)_1(\cdot,\cdot,0) \right\|^2_{L^2(\omega)} \leq \frac{c\varepsilon^2}{r^3} \mathcal{E}_{\Omega^-}(u^{\varepsilon,r}). \tag{4.24}$$

Indeed, we have the same estimates on $\left(\mathcal{R}^{\varepsilon,r}\right)_2(0)$ and $\left(\mathcal{R}^{\varepsilon,r}\right)_3(0)$ in $L^2(\omega)$, so that

$$\left\| \mathcal{R}^{\varepsilon,r}(\cdot,\cdot,0) \right\|^2_{(L^2(\omega))^3} \leq \frac{c\varepsilon^2}{r^3} \mathcal{E}_{\Omega^-}(u^{\varepsilon,r}). \tag{4.25}$$

To obtain an estimate on $\mathcal{U}^{\varepsilon,r}(\cdot,\cdot,0)$, we just write, that for any $(p,q)\in\mathcal{N}^\varepsilon$,

$$\mathcal{U}^{\varepsilon,r}_{pq}(0) = \frac{1}{\pi r^2} \int_{\mathcal{D}^{\varepsilon,r}_{pq}} u^{\varepsilon,r}(x_1,x_2,0)\,dx_1dx_2, \tag{4.26}$$

and then by Cauchy-Schwarz's inequality

$$\left| \mathcal{U}^{\varepsilon,r}_{pq}(0) \right|^2 \leq \frac{c}{r^2} \int_{\mathcal{D}^{\varepsilon,r}_{pq}} |u^{\varepsilon,r}(x_1,x_2,0)|^2 dx_1dx_2. \tag{4.27}$$

Due to the definition (4.2)÷(4.4) of $\mathcal{U}^{\varepsilon,r}$, summing up with respect to $(p,q)\in\mathcal{N}^\varepsilon$, we obtain

$$\left\| \mathcal{U}^{\varepsilon,r}(\cdot,\cdot,0) \right\|^2_{(L^2(\omega))^3} \leq \frac{c\varepsilon^2}{r^2} \left\| u^{\varepsilon,r}(\cdot,\cdot,0) \right\|^2_{(L^2(\omega))^3}.$$

Now, again with the help of the Korn's inequality in $\Omega^-$ (see again (3.15)) and of the trace theorem in $\Omega^-$, it yields

$$\left\| \mathcal{U}^{\varepsilon,r}(\cdot,\cdot,0) \right\|^2_{(L^2(\omega))^3} \leq \frac{c\varepsilon^2}{r^2} \mathcal{E}_{\Omega^-}(u^{\varepsilon,r}). \tag{4.28}$$

Step 2. Estimates on $\mathcal{U}^{\varepsilon,r}$ and $\mathcal{R}^{\varepsilon,r}$.

For any $(p,q) \in \mathcal{N}^\varepsilon$, recall that by (3.12)

$$\left\| \frac{d\mathcal{R}^{\varepsilon,r}_{pq}}{dx_3} \right\|^2_{(L^2(]0,L[))^3} \leq \frac{c}{r^4} \mathcal{E}_{\mathcal{P}^{\varepsilon,r}_{pq}}(u^{\varepsilon,r}).$$

Then, with the help of (4.8), we deduce that

$$\left\| \frac{\partial\mathcal{R}^{\varepsilon,r}}{\partial x_3} \right\|^2_{(L^2(\Omega^+))^3} \leq \frac{c\varepsilon^2}{r^4} \mathcal{E}_{\Omega^+_{\varepsilon,r}}(u^{\varepsilon,r}), \tag{4.29}$$

which, together with (4.25) permits to obtain

$$\|\mathcal{R}^{\varepsilon,r}\|^2_{(L^2(\omega,H^1(]0,L[)))^3} \leq \frac{c\varepsilon^2}{r^4} \mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r}), \tag{4.30}$$

since $\mathcal{E}_{\Omega^+_{\varepsilon,r}}(u^{\varepsilon,r}) + \mathcal{E}_{\Omega^-}(u^{\varepsilon,r}) = \mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r})$ (the sharper estimate (4.25) will be used in Subsection 5.5).

To obtain estimates on $\mathcal{U}^{\varepsilon,r}$, we first investigate the components $\mathcal{U}^{\varepsilon,r}_1$ and $\mathcal{U}^{\varepsilon,r}_2$, and we only give the proof for $\mathcal{U}^{\varepsilon,r}_1$ (since it is identical for $\mathcal{U}^{\varepsilon,r}_2$).

Due to (3.11), for any $(p,q) \in \mathcal{N}^\varepsilon$, we have that

$$\left\| \frac{d\left(\mathcal{U}^{\varepsilon,r}_{pq}\right)_1}{dx_3} \right\|^2_{L^2(]0,L[)} \leq c\left[ \left\| \left(\mathcal{R}^{\varepsilon,r}_{pq}\right)_2 \right\|^2_{L^2(]0,L[)} + \frac{1}{r^2} \mathcal{E}_{\mathcal{P}^{\varepsilon,r}_{pq}}(u^{\varepsilon,r}) \right],$$

from which, by using (4.7), it follows that

$$\left\| \frac{\partial\mathcal{U}^{\varepsilon,r}_1}{\partial x_3} \right\|^2_{L^2(\Omega^+)} \leq c\left[ \|\mathcal{R}^{\varepsilon,r}_2\|^2_{L^2(\Omega^+)} + \frac{\varepsilon^2}{r^2} \mathcal{E}_{\Omega^+_{\varepsilon,r}}(u^{\varepsilon,r}) \right],$$

where $c$ is a constant independent of $\varepsilon$. Then, with the help of (4.30), we obtain that (since $r << 1$)

$$\left\| \frac{\partial\mathcal{U}^{\varepsilon,r}_1}{\partial x_3} \right\|^2_{L^2(\Omega^+)} \leq c\frac{\varepsilon^2}{r^4} \mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r}). \tag{4.31}$$

In view of (4.28), we deduce that

$$\|\mathcal{U}^{\varepsilon,r}_1\|^2_{L^2(\Omega^+)} \leq c\frac{\varepsilon^2}{r^4} \mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r}). \tag{4.32}$$

Similarly we have

$$\|\mathcal{U}^{\varepsilon,r}_2\|^2_{L^2(\Omega^+)} \leq c\frac{\varepsilon^2}{r^4} \mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r}), \tag{4.33}$$

$$\left\| \frac{\partial\mathcal{U}^{\varepsilon,r}_2}{\partial x_3} \right\|^2_{L^2(\Omega^+)} \leq c\frac{\varepsilon^2}{r^4} \mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r}). \tag{4.34}$$

Let us now consider $\mathcal{U}_3^{\varepsilon,r}$. For any $(p,q) \in \mathcal{N}^\varepsilon$, we have from (3.11)

$$\left\| \frac{d\left(\mathcal{U}_{pq}^{\varepsilon,r}\right)_3}{dx_3} \right\|_{L^2(]0,L[)}^2 \leq \frac{c}{r^2} \mathcal{E}_{\mathcal{P}_{pq}^{\varepsilon,r}}(u^{\varepsilon,r}),$$

which yields with (4.7)

$$\left\| \frac{\partial \mathcal{U}_3^{\varepsilon,r}}{\partial x_3} \right\|_{L^2(\Omega^+)}^2 \leq c\frac{\varepsilon^2}{r^2} \mathcal{E}_{\Omega_{\varepsilon,r}^+}(u^{\varepsilon,r}).$$

By using (4.28), it follows that

$$\|\mathcal{U}_3^{\varepsilon,r}\|_{L^2(\Omega^+)}^2 \leq c\frac{\varepsilon^2}{r^2} \mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r}). \tag{4.35}$$

## 4.2 Uniform bound on $\overline{u}^{\varepsilon,r}$ in term of $\mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r})$

Let us recall that in view of (3.13)-(3.14) and of the definition (4.12) of $\overline{u}^{\varepsilon,r}$, one has for any $(p,q) \in \mathcal{N}^\varepsilon$,

$$\|\overline{u}^{\varepsilon,r}\|_{\left(L^2(\mathcal{P}_{pq}^{\varepsilon,r})\right)^3}^2 \leq cr^2 \mathcal{E}_{\mathcal{P}_{pq}^{\varepsilon,r}}(u^{\varepsilon,r}),$$

and

$$\|D\overline{u}^{\varepsilon,r}\|_{\left(L^2(\mathcal{P}_{pq}^{\varepsilon,r})\right)^9}^2 \leq c \mathcal{E}_{\mathcal{P}_{pq}^{\varepsilon,r}}(u^{\varepsilon,r}).$$

Through summation over $(p,q) \in \mathcal{N}^\varepsilon$, we deduce that

$$\|\overline{u}^{\varepsilon,r}\|_{\left(L^2(\Omega_{\varepsilon,r}^+)\right)^3}^2 \leq cr^2 \mathcal{E}_{\Omega_{\varepsilon,r}^+}(u^{\varepsilon,r}), \tag{4.36}$$

and

$$\|D\overline{u}^{\varepsilon,r}\|_{\left(L^2(\Omega_{\varepsilon,r}^+)\right)^9}^2 \leq c \mathcal{E}_{\Omega_{\varepsilon,r}^+}(u^{\varepsilon,r}). \tag{4.37}$$

## 4.3 Estimates on $u^{\varepsilon,r}$ in term of $\mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r})$

First recall that from (3.7) and (4.12), we have, for any $(p,q) \in \mathcal{N}^\varepsilon$, and for almost every $(x_1,x_2,x_3) \in \Omega_{\varepsilon,r}^+$

$$\begin{aligned} &u_1^{\varepsilon,r}(x_1,x_2,x_3) = \\ &\left(\mathcal{U}_{pq}^{\varepsilon,r}\right)_1(x_3) - \left(\mathcal{R}_{pq}^{\varepsilon,r}\right)_3(x_3)(x_2-\varepsilon q) + \overline{u}_1^{\varepsilon,r}(x_1,x_2,x_3), \text{ if } (x_1,x_2) \in \mathcal{D}_{pq}^{\varepsilon,r}. \end{aligned} \tag{4.38}$$

$$\begin{aligned} &u_2^{\varepsilon,r}(x_1,x_2,x_3) = \\ &\left(\mathcal{U}_{pq}^{\varepsilon,r}\right)_2(x_3) + \left(\mathcal{R}_{pq}^{\varepsilon,r}\right)_3(x_3)(x_1-\varepsilon p) + \overline{u}_2^{\varepsilon,r}(x_1,x_2,x_3), \text{ if } (x_1,x_2) \in \mathcal{D}_{pq}^{\varepsilon,r}. \end{aligned} \tag{4.39}$$

We derive first $L^2$ estimates on $u_1^{\varepsilon,r}$ (the details are identical for $u_2^{\varepsilon,r}$). We have, for any $(p,q)\in\mathcal{N}^\varepsilon$ and for almost every $x_3\in]0,L[$

$$\int_{\mathcal{D}_{pq}^{\varepsilon,r}}|u_1^{\varepsilon,r}(x_1,x_2,x_3)|^2dx_1dx_2\leq$$

$$c\left[r^2\left|\left(\mathcal{U}_{pq}^{\varepsilon,r}\right)_1(x_3)\right|^2+r^4\left|\left(\mathcal{R}_{pq}^{\varepsilon,r}\right)_3(x_3)\right|^2+\int_{\mathcal{D}_{pq}^{\varepsilon,r}}|\overline{u}_1^{\varepsilon,r}(x_1,x_2,x_3)|^2dx_1dx_2\right].$$

By adding the previous inequalities with respect to $(p,q)\in\mathcal{N}^\varepsilon$, and by integrating over $]0,L[$, we obtain, in view of (4.5) and (4.6)

$$\|u_1^{\varepsilon,r}\|^2_{L^2(\Omega^+_{\varepsilon,r})}\leq c\left[\frac{r^2}{\varepsilon^2}\|\mathcal{U}_1^{\varepsilon,r}\|^2_{L^2(\Omega^+)}+\frac{r^4}{\varepsilon^2}\|\mathcal{R}_3^{\varepsilon,r}\|^2_{L^2(\Omega^+)}+\|\overline{u}_1^{\varepsilon,r}\|^2_{L^2(\Omega^+_{\varepsilon,r})}\right].$$

Appealing now to (4.30), (4.32) and (4.36), it yields that

$$\|u_1^{\varepsilon,r}\|^2_{L^2(\Omega^+_{\varepsilon,r})}\leq c\left[\frac{1}{r^2}+1+r^2\right]\mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r}).$$

Finally, and proceeding identically for $u_2^{\varepsilon,r}$, we obtain

$$\|u_\alpha^{\varepsilon,r}\|^2_{L^2(\Omega^+_{\varepsilon,r})}\leq\frac{c}{r^2}\mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r}),\text{ for }\alpha=1,2. \tag{4.40}$$

As far as $u_3^{\varepsilon,r}$ is concerned, recall that with (3.7) and (4.12) we have, for any $(p,q)\in\mathcal{N}^\varepsilon$, and for almost every $(x_1,x_2,x_3)\in\Omega^+_{\varepsilon,r}$

$$\begin{aligned}&u_3^{\varepsilon,r}(x_1,x_2,x_3)=\left(\mathcal{U}_{pq}^{\varepsilon,r}\right)_3(x_3)+\left(\mathcal{R}_{pq}^{\varepsilon,r}\right)_1(x_3)(x_2-\varepsilon q)-\\&\left(\mathcal{R}_{pq}^{\varepsilon,r}\right)_2(x_3)(x_1-\varepsilon p)+\overline{u}_3^{\varepsilon,r}(x_1,x_2,x_3),\text{ if }(x_1,x_2)\in\mathcal{D}_{pq}^{\varepsilon,r}.\end{aligned} \tag{4.41}$$

This implies that for any $(p,q)\in\mathcal{N}^\varepsilon$ and for almost every $x_3\in]0,L[$

$$\int_{\mathcal{D}_{pq}^{\varepsilon,r}}|u_3^{\varepsilon,r}(x_1,x_2,x_3)|^2dx_1dx_2\leq$$

$$c\left[r^2\left|\left(\mathcal{U}_{pq}^{\varepsilon,r}\right)_3(x_3)\right|^2+r^4\left(\left|\left(\mathcal{R}_{pq}^{\varepsilon,r}\right)_1(x_3)\right|^2+\left|\left(\mathcal{R}_{pq}^{\varepsilon,r}\right)_2(x_3)\right|^2\right)+\int_{\mathcal{D}_{pq}^{\varepsilon,r}}|\overline{u}_3^{\varepsilon,r}(x_1,x_2,x_3)|^2dx_1dx_2\right].$$

Proceeding as for $u_1^{\varepsilon,r}$, it yields with the help of (4.5) and (4.6)

$$\|u_3^{\varepsilon,r}\|^2_{L^2(\Omega^+_{\varepsilon,r})}\leq c\left[\frac{r^2}{\varepsilon^2}\|\mathcal{U}_3^{\varepsilon,r}\|^2_{L^2(\Omega^+)}+\frac{r^4}{\varepsilon^2}\left(\|\mathcal{R}_1^{\varepsilon,r}\|^2_{L^2(\Omega^+)}+\|\mathcal{R}_2^{\varepsilon,r}\|^2_{L^2(\Omega^+)}\right)+\|\overline{u}_3^{\varepsilon,r}\|^2_{L^2(\Omega^+_{\varepsilon,r})}\right].$$

Now we use (4.30), (4.35) and (4.36) to obtain

$$\|u_3^{\varepsilon,r}\|^2_{L^2(\Omega^+_{\varepsilon,r})}\leq c\left[1+r^2\right]\mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r}),$$

and finally

$$\|u_3^{\varepsilon,r}\|^2_{L^2(\Omega^+_{\varepsilon,r})}\leq c\mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r}). \tag{4.42}$$

## 4.4 *A priori* estimates on $u^{\varepsilon,r}$

The inserting (2.13) into (3.1) leads to

$$\mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r}) \le \sum_{\alpha=1}^{2} \|f_\alpha^{\varepsilon,r}\|_{L^2(\Omega^+_{\varepsilon,r})}\|u_\alpha^{\varepsilon,r}\|_{L^2(\Omega^+_{\varepsilon,r})} + \|f_3^{\varepsilon,r}\|_{L^2(\Omega^+_{\varepsilon,r})}\|u_3^{\varepsilon,r}\|_{L^2(\Omega^+_{\varepsilon,r})} +$$

$$\sum_{i=1}^{3} \|f_i^{\varepsilon,r}\|_{L^2(\Omega^-)}\|u_i^{\varepsilon,r}\|_{L^2(\Omega^-)}.$$

Then the estimates on $\|u_i^{\varepsilon,r}\|^2_{L^2(\Omega^+_{\varepsilon,r})}$ in the previous section and estimates (3.15) on $\|u^{\varepsilon,r}\|^2_{(L^2(\Omega^-))^3}$ permit to obtain

$$\begin{aligned}&\mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r}) \le \\ &c\left[\frac{1}{r}\sum_{\alpha=1}^{2}\|f_\alpha^{\varepsilon,r}\|_{L^2(\Omega^+_{\varepsilon,r})} + \|f_3^{\varepsilon,r}\|_{L^2(\Omega^+_{\varepsilon,r})} + \sum_{i=1}^{3}\|f_i^{\varepsilon,r}\|_{L^2(\Omega^-)}\right]\left(\mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r})\right)^{\frac{1}{2}}.\end{aligned} \tag{4.43}$$

In view of (4.43), the assumptions (2.19)÷(2.21) on the forces $f^{\varepsilon,r}$ in $\Omega^+_{\varepsilon,r}$ and $\Omega^-$ appear (*a posteriori*) natural to obtain an estimate on $\mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r})$, namely here

$$\mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r}) \le c. \tag{4.44}$$

**Remark 4.1.** *Indeed, Problem (2.11)÷(2.16) is linear with respect to $f^{\varepsilon,r}$. Then at the possible rescaling of $u^{\varepsilon,r}$, what is important in (4.43) is the relative behavior between $f_\alpha^{\varepsilon,r}$ and $f_3^{\varepsilon,r}$ in $\Omega^+_{\varepsilon,r}$ and $f_i^{\varepsilon,r}$ in $\Omega^-$. Here we have decided to normalize $f_i^{\varepsilon,r}$ in $\Omega^-$, to obtain an elastic energy $\mathcal{E}_{\Omega_{\varepsilon,r}}(u^{\varepsilon,r})$ of order 1 with respect to $\varepsilon$.*

Once (4.44) is established, the estimates stated in the following lemma are direct consequences of the previous sections.

**Lemma 4.2.** *Under assumptions (2.19)÷ (2.21), there exists a constant $c$ independent of $\varepsilon$ and $r$ such that*

$$r\|u_\alpha^{\varepsilon,r}\|_{L^2(\Omega^+_{\varepsilon,r})} \le c, \text{ for } \alpha = 1,2, \tag{4.45}$$

$$\|u_3^{\varepsilon,r}\|_{L^2(\Omega^+_{\varepsilon,r})} \le c, \tag{4.46}$$

$$\|u_i^{\varepsilon,r}\|_{L^2(\Omega^-)} \le c, \text{ for } i = 1,2,3, \tag{4.47}$$

$$\|\gamma_{ij}(u^{\varepsilon,r})\|_{L^2(\Omega^+_{\varepsilon,r})} \le c, \text{ for } i,j = 1,2,3 \tag{4.48}$$

$$\|\gamma_{ij}(u^{\varepsilon,r})\|_{L^2(\Omega^-)} \le c, \text{ for } i,j = 1,2,3, \tag{4.49}$$

$$\frac{r^2}{\varepsilon}\|\mathcal{U}_\alpha^{\varepsilon,r}\|_{L^2(\omega,H^1(]0,L[))} \le c, \text{ for } \alpha = 1,2, \tag{4.50}$$

$$\frac{r}{\varepsilon}\|\mathcal{U}_3^{\varepsilon,r}\|_{L^2(\omega,H^1(]0,L[))} \le c, \tag{4.51}$$

$$\frac{r^2}{\varepsilon} \|\mathcal{R}_i^{\varepsilon,r}\|_{L^2(\omega,H^1(]0,L[))} \leq c, \ \ for \ i = 1,2,3, \tag{4.52}$$

$$\frac{r}{\varepsilon} \left\| \frac{\partial \mathcal{U}^{\varepsilon,r}}{\partial x_3} - (\mathcal{R}^{\varepsilon,r} \wedge e_3) \right\|_{(L^2(\Omega^+))^3} \leq c, \tag{4.53}$$

$$\|\overline{u}^{\varepsilon,r}\|_{\left(L^2(\Omega^+_{\varepsilon,r})\right)^3} \leq cr, \tag{4.54}$$

$$\|D\overline{u}^{\varepsilon,r}\|_{\left(L^2(\Omega^+_{\varepsilon,r})\right)^9} \leq c. \tag{4.55}$$

# 5 Unfolding operator and estimates on the unfold fields

In the sequel of this paper, $\{\varepsilon\}$ will be a sequence of positive real numbers which tends to zero and the radius of the rods will take values in a sequence $\{r_\varepsilon\}_\varepsilon$ which also tends to zero. For sake of simplicity, we will drop the index $r_\varepsilon$ in the notations

In this section we first adapt the notion of "unfolding technique", introduced in [8] for thin or periodic structures, to take into account both the usual rescaling in rods theory and the periodic character of $\Omega^+_\varepsilon$. References on unfolding operators can be found in [8], [11] and [15]. Then we deduce from Section 4.4, the estimates on the unfolded various quantities studied in this section.

## 5.1 The unfolding operator

Throughout the paper $D$ will now denote the unit disk of $\mathbb{R}^2$: $D = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$.

Let $v$ be a function of $L^2(\Omega^+_\varepsilon)$. We define the function $\mathcal{T}^\varepsilon(v)$ on $\Omega^+ \times D$ by, for almost $(x_1,x_2,x_3) \in \Omega^+$ and $(X_1,X_2) \in D$,

$$\mathcal{T}^\varepsilon(v)(x_1,x_2,x_3,X_1,X_2) =$$

$$\begin{cases} v(p\varepsilon + r_\varepsilon X_1, q\varepsilon + r_\varepsilon X_2, x_3), \\ \qquad \text{if } (x_1,x_2) \in \left]\varepsilon p - \dfrac{\varepsilon}{2}, \varepsilon p + \dfrac{\varepsilon}{2}\right[ \times \left]\varepsilon q - \dfrac{\varepsilon}{2}, \varepsilon q + \dfrac{\varepsilon}{2}\right[, \quad (p,q) \in \mathcal{N}_\varepsilon, \\ 0, \qquad \text{if } (x_1,x_2) \in \omega \setminus \widetilde{\omega}_\varepsilon \end{cases} \tag{5.1}$$

(recall that $\widetilde{\omega}_\varepsilon$ is defined in (4.1)).

Let us make a few comments on this definition. First, it is clear that $x_3$ appears in 5.1 as a parameter. Then $\mathcal{T}^\varepsilon(v)$ is well defined on $\Omega^+ \times D$ since for $(X_1,X_2) \in D$, one has $(\varepsilon p + r_\varepsilon X_1, \varepsilon q + r_\varepsilon X_2, x_3) \in \mathcal{P}^\varepsilon_{pq}$. For the points $(x_1,x_2,x_3) \in \Omega^+$ for which $(x_1,x_2) \in \omega \setminus \widetilde{\omega}_\varepsilon$, $\mathcal{T}^\varepsilon(v)(x_1,x_2,x_3,X_1,X_2) = 0$ a.e.. The main interest in considering $\mathcal{T}^\varepsilon(v)$ rather than $v$, is that the effect of the oscillations of $\Omega^+_\varepsilon$ is, in some sense, decoupled to the slow (and here disconnected) variation of $(x_1,x_2)$. Namely, $(x_1,x_2)$ are split into $(\varepsilon p, \varepsilon q)$ in one hand and $(X_1,X_2)$ on the other hand.

As a convention, if $v \in L^2(\Omega^+)$, we set $\mathcal{T}^\varepsilon(v) = \mathcal{T}^\varepsilon(v_{|_{\Omega^+_\varepsilon}})$.

The following lemma contains the main properties of the operator $\mathcal{T}^\varepsilon$ which will be used throughout the paper.

**Lemma 5.1.**

*(a) For all function $v$ and $w$ in $L^2(\Omega_\varepsilon^+)$, one has*

$$\int_{\Omega_\varepsilon^+} vw dx_1 dx_2 dx_3 = \frac{r_\varepsilon^2}{\varepsilon^2} \int_{\Omega^+ \times D} \mathcal{T}^\varepsilon(v) \mathcal{T}^\varepsilon(w) dx_1 dx_2 dx_3 dX_1 dX_2.$$

*(b) In the case $r_\varepsilon = k\varepsilon$, for any function $v$ in $L^2(\Omega^+)$,*

$$\mathcal{T}^\varepsilon(v) \to v \text{ strongly in } L^2(\Omega^+ \times D),$$

*as $\varepsilon$ tends to 0.*

*(c) In the case where $\dfrac{r_\varepsilon}{\varepsilon}$ tends to zero, and for any function $v \in C^0(\overline{\Omega^+})$,*

$$\mathcal{T}^\varepsilon(v) \to v \text{ strongly in } L^2(\Omega^+ \times D),$$

*as $\varepsilon$ tends to 0.*

*(d) In the case $r_\varepsilon = k\varepsilon$, if $\{v_\varepsilon\}_\varepsilon$ is a sequence of $L^2(\Omega^+)$ such that $v_\varepsilon \to v$ strongly in $L^2(\Omega^+)$, then*

$$\mathcal{T}^\varepsilon(v_\varepsilon) \to v \text{ strongly in } L^2(\Omega^+ \times D),$$

*as $\varepsilon$ tends to 0.*

*(e) For any $v \in H^1(\Omega_\varepsilon^+)$,*

$$\frac{\partial(\mathcal{T}^\varepsilon(v))}{\partial X_\alpha} = r_\varepsilon \mathcal{T}^\varepsilon\left(\frac{\partial v}{\partial x_\alpha}\right) \text{ a.e. in } \Omega^+ \times D, \text{ for } \alpha = 1, 2,$$

*and*

$$\frac{\partial(\mathcal{T}^\varepsilon(v))}{\partial x_3} = \mathcal{T}^\varepsilon\left(\frac{\partial v}{\partial x_3}\right) \text{ a.e. in } \Omega^+ \times D.$$

*Proof.* In order to obtain (a) we write

$$\int_{\Omega_\varepsilon^+} vw dx_1 dx_2 dx_3 = \int_0^L \sum_{(p,q)\in\mathcal{N}^\varepsilon} \int_{\mathcal{D}_{pq}^\varepsilon} v(x_1, x_2, x_3) w(x_1, x_2, x_3) dx_1 dx_2 dx_3 =$$

$$r_\varepsilon^2 \int_0^L \sum_{(p,q)\in\mathcal{N}^\varepsilon} \int_D v(\varepsilon p + r_\varepsilon X_1, \varepsilon q + r_\varepsilon X_2, x_3) w(\varepsilon p + r_\varepsilon X_1, \varepsilon q + r_\varepsilon X_2, x_3) dX_1 dX_2 dx_3 =$$

$$r_\varepsilon^2 \int_0^L \sum_{(p,q)\in\mathcal{N}^\varepsilon} \frac{1}{\varepsilon^2} \int_{D\times]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}[\times]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}[} v(\varepsilon p + r_\varepsilon X_1, \varepsilon q + r_\varepsilon X_2, x_3)$$

$$w(\varepsilon p + r_\varepsilon X_1, \varepsilon q + r_\varepsilon X_2, x_3) dX_1 dX_2 dx_1 dx_2 dx_3 =$$

$$\frac{r_\varepsilon^2}{\varepsilon^2} \int_{]0,L[\times D\times\omega} \mathcal{T}^\varepsilon(v) \mathcal{T}^\varepsilon(w) dx_3 dX_1 dX_2 dx_1 dx_2 =$$

$$\frac{r_\varepsilon^2}{\varepsilon^2} \int_{\Omega^+ \times D} \mathcal{T}^\varepsilon(v) \mathcal{T}^\varepsilon(w) dx_1 dx_2 dx_3 dX_1 dX_2.$$

The last equality being due to $\mathcal{T}^\varepsilon(v) = 0$ if $(x_1, x_2) \in \omega \setminus \widetilde{\omega}_\varepsilon$.

To prove (b) and (c), first consider a function $\varphi \in C^0(\overline{\Omega^+})$. By definition (5.1) of $\mathcal{T}^\varepsilon$, we have for any $(x_1, x_2, x_3) \in \Omega^+$ and $(X_1, X_2) \in D$,

$$|\mathcal{T}^\varepsilon(\varphi)(x_1, x_2, x_3, X_1, X_2) - \varphi(x_1, x_2, x_3)| = |\varphi(\varepsilon p + r_\varepsilon X_1, \varepsilon q + r_\varepsilon X_2, x_3) - \varphi(x_1, x_2, x_3)|\,,$$

if $(x_1, x_2) \in \left]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\right[ \times \left]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\right[$ and $(p, q) \in \mathcal{N}_\varepsilon$,

$$|\mathcal{T}^\varepsilon(\varphi)(x_1, x_2, x_3, X_1, X_2) - \varphi(x_1, x_2, x_3)| = |\varphi(x_1, x_2, x_3)|\,,$$

if $(x_1, x_2) \in \omega \setminus \widetilde{\omega}_\varepsilon$.
Then, since $\varphi \in C^0(\overline{\Omega^+})$,

$$|\mathcal{T}^\varepsilon(\varphi)(x_1, x_2, x_3, X_1, X_2) - \varphi(x_1, x_2, x_3)| \leq \delta(\varepsilon)\chi_{\widetilde{\omega}_\varepsilon} + (1 - \chi_{\widetilde{\omega}_\varepsilon})\, \|\varphi\|_{C^0(\overline{\Omega^+})}\,, \tag{5.2}$$

where $\delta(\varepsilon)$ tends to zero as $\varepsilon$ tends to zero, and $\chi_{\widetilde{\omega}\varepsilon}$ denotes the characteristic function of $\widetilde{\omega}_\varepsilon$. It follows that

$$\|\mathcal{T}^\varepsilon(\varphi) - \varphi\|_{L^2(\Omega^+ \times D)} \leq c\delta(\varepsilon) + c(meas(\omega - \widetilde{\omega}_\varepsilon))^{\frac{1}{2}} \|\varphi\|_{C^0(\overline{\Omega^+})} \tag{5.3}$$

Now when $\varepsilon$ tends to 0, $meas(\omega - \widetilde{\omega}_\varepsilon)$ tends to zero, because $\partial\omega$ is assumed to be Lipschitz and $\varepsilon \to 0$, so that we obtain

$$\mathcal{T}^\varepsilon(v) \to v \text{ strongly in } L^2(\Omega^+ \times D), \tag{5.4}$$

as $\varepsilon$ tends to 0. This establish (c).

To obtain (b), remark that if $r_\varepsilon = k\varepsilon$, (a), gives

$$\|\mathcal{T}^\varepsilon(\varphi) - \mathcal{T}^\varepsilon(\psi)\|_{L^2(\Omega^+ \times D)} = \frac{1}{k}\, \|\varphi - \psi\|_{L^2(\Omega^+_\varepsilon)} \leq \frac{1}{k}\, \|\varphi - \psi\|_{L^2(\Omega^+)}\,, \tag{5.5}$$

for all $\varphi$ and $\psi$ in $L^2(\Omega^+)$. In view of (5.4) and (5.5), a classical density argument shows that (b) hods true. Property (d) is an easy consequence of (b) and of (5.5). Property (e) follows from the standard chain rule formulae in each cell $\left]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\right[ \times \left]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\right[$ and it is trivial if $(x_1, x_2) \in \omega \setminus \widetilde{\omega}_\varepsilon$. □

**Remark 5.2.** *Let us conclude this section with a remark which will be useful to identify the junction condition between $\Omega^+$ and $\Omega^-$. Consider a function $v \in L^2(\Omega_\varepsilon)$. Then, since $x_3$ appears as a parameter in (5.1), one can also define $\mathcal{T}^\varepsilon(v)$ in $\Omega^- \times D$ (i.e. for $-l < x_3 < 0$). In the case where $r_\varepsilon = k\varepsilon$ and if now $\{v_\varepsilon\}_\varepsilon \subset L^2(\Omega_\varepsilon)$ is a sequence such that $v_{\varepsilon|_{\Omega^-}}$ converges strongly in $L^2(\Omega^-)$ to a function $v \in L^2(\Omega^-)$, as $\varepsilon \to 0$, then*

$$\mathcal{T}^\varepsilon(v_\varepsilon) \to v \textit{ strongly in } L^2(\Omega^- \times D), \tag{5.6}$$

*as $\varepsilon$ tends to 0.*

## 5.2 Estimates on the unfold fields

Lemma 4.2 and Lemma 5.1 together with 4.38, 4.39, 4.41 permit to obtain the following Lemma:

**Lemma 5.3.** *Under assumptions (2.19)÷(2.21), there exists a constant $c$ independent of $\varepsilon$ such that*

$$r_\varepsilon \, \|\mathcal{T}^\varepsilon(u^\varepsilon_\alpha)\|_{L^2(\omega,H^1(D\times]0,L[))} \le c\big(1+\frac{\varepsilon}{r_\varepsilon}\big), \ \textit{for } \alpha=1,2, \tag{5.7}$$

$$\|\mathcal{T}^\varepsilon(u^\varepsilon_3)\|_{L^2(\omega,H^1(D\times]0,L[))} \le c\big(1+\frac{\varepsilon}{r_\varepsilon}\big), \tag{5.8}$$

$$\frac{r_\varepsilon}{\varepsilon} \, \|\mathcal{T}^\varepsilon(\gamma_{ij}(u^\varepsilon))\|_{L^2(\Omega^+\times D)} \le c, \ \textit{for } i,j=1,2,3 \tag{5.9}$$

$$\frac{1}{\varepsilon} \, \|\mathcal{T}^\varepsilon(\overline{u}^\varepsilon)\|_{(L^2(\Omega^+\times D))^3} \le c, \tag{5.10}$$

$$\begin{cases} \dfrac{1}{\varepsilon}\left\|\dfrac{\partial(\mathcal{T}^\varepsilon(\overline{u}^\varepsilon))}{\partial X_\alpha}\right\|_{(L^2(\Omega^+\times D))^3} \le c, \ \textit{for } \alpha=1,2, \\[2ex] \dfrac{r_\varepsilon}{\varepsilon}\left\|\dfrac{\partial(\mathcal{T}^\varepsilon(\overline{u}^\varepsilon))}{\partial x_3}\right\|_{(L^2(\Omega^+\times D))^3} \le c, \end{cases} \tag{5.11}$$

$$\frac{r_\varepsilon}{\varepsilon} \, \big\|\mathcal{T}^\varepsilon(\sigma^\varepsilon_{ij})\big\|_{L^2(\Omega^+\times D)} \le c, \ \textit{for } i,j=1,2,3. \tag{5.12}$$

Until now, we have kept the possibility in all the above estimates that $r_\varepsilon$ and $\varepsilon$ may behave in a way such that $\lim_{\varepsilon\to 0}\frac{r_\varepsilon}{\varepsilon}=k$, where $k$ is a real number such that $0\le k<\frac{1}{2}$. Actually, here we have to distinguish the case where $\frac{r_\varepsilon}{\varepsilon}=k>0$ to the case where $\lim_{\varepsilon\to 0}\frac{r_\varepsilon}{\varepsilon}=0$. We first investigate in the following the case where $r_\varepsilon=k\varepsilon$, and postpone the analysis for the case $\lim_{\varepsilon\to 0}\frac{r_\varepsilon}{\varepsilon}=0$ to Section 7.

## 5.3 Weak limits of the fields (case $r_\varepsilon = k\varepsilon$)

As explained above, we assume here that $r_\varepsilon=k\varepsilon$ and we just introduce the notations for the weak limit, up to a subsequence still denoted by $\varepsilon$, of the bounded fields appearing in Lemma 4.2 and Lemma 5.3.

**Lemma 5.4.** *Assume (2.19)÷(2.21), and that $r_\varepsilon=k\varepsilon$.*

*For a subsequence, still denoted by $\{\varepsilon\}$,*

- *there exist $u^0_i\in L^2(\omega,H^1(D\times]0,L[))$ and $\overline{u}^0_i\in L^2(\Omega^+,H^1(D))$, for $i=1,2,3$, such that, as $\varepsilon$ tends to zero,*

$$\varepsilon\mathcal{T}^\varepsilon(u^\varepsilon_\alpha)\rightharpoonup u^0_\alpha \ \textit{weakly in } L^2(\omega,H^1(D\times]0,L[)), \ \textit{for } \alpha=1,2, \tag{5.13}$$

$$\mathcal{T}^\varepsilon(u^\varepsilon_3)\rightharpoonup u^0_3 \ \textit{weakly in } L^2(\omega,H^1(D\times]0,L[)), \tag{5.14}$$

$$\frac{1}{\varepsilon}\mathcal{T}^\varepsilon(\overline{u}^\varepsilon_i)\rightharpoonup \overline{u}^0_i \ \textit{weakly in } L^2(\Omega^+,H^1(D)), \ \textit{for } i=1,2,3; \tag{5.15}$$

• *there exist* $\mathcal{U}_i^0 \in L^2(\omega, H^1(]0,L[))$, $\mathcal{R}_i^0 \in L^2(\omega, H^1(]0,L[))$, *for* $i = 1,2,3$, *and* $Z \in (L^2(\Omega^+))^3$ *such that, as* $\varepsilon$ *tends to zero,*

$$\varepsilon\mathcal{U}_\alpha^\varepsilon \rightharpoonup \mathcal{U}_\alpha^0 \text{ weakly in } L^2\left(\omega, H^1(]0,L[)\right), \text{ for } \alpha = 1,2, \tag{5.16}$$

$$\mathcal{U}_3^\varepsilon \rightharpoonup \mathcal{U}_3^0 \text{ weakly in } L^2\left(\omega, H^1(]0,L[)\right), \tag{5.17}$$

$$\varepsilon\mathcal{R}_i^\varepsilon \rightharpoonup \mathcal{R}_i^0 \text{ weakly in } L^2\left(\omega, H^1(]0,L[)\right), \text{ for } i = 1,2,3, \tag{5.18}$$

$$\frac{\partial \mathcal{U}^\varepsilon}{\partial x_3} - (\mathcal{R}^\varepsilon \wedge e_3) \rightharpoonup Z \text{ weakly in } \left(L^2(\Omega^+)\right)^3; \tag{5.19}$$

• *there exist* $X_{ij} \in L^2(\Omega^+ \times D)$ *and* $\Sigma_{ij} \in L^2(\Omega^+ \times D)$, *for* $i,j = 1,2,3$, *such that, as* $\varepsilon$ *tends to zero,*

$$\mathcal{T}^\varepsilon(\gamma_{ij}(u^\varepsilon)) \rightharpoonup X_{ij} \text{ weakly in } L^2(\Omega^+ \times D), \text{ for } i,j = 1,2,3, \tag{5.20}$$

$$\mathcal{T}^\varepsilon(\sigma_{ij}^\varepsilon) \rightharpoonup \Sigma_{ij} \text{ weakly in } L^2(\Omega^+ \times D), \text{ for } i,j = 1,2,3; \tag{5.21}$$

• *there exist* $u_i^- \in H^1(\Omega^-)$, *with* $u_i^- = 0$ *on* $\partial\omega\times]-l,0[$, *for* $i = 1,2,3$, *such that, as* $\varepsilon$ *tends to zero,*

$$u_i^\varepsilon \rightharpoonup u_i^- \text{ weakly in } H^1(\Omega^-), \text{ strongly in } L^2(\Omega^-). \tag{5.22}$$

## 5.4 Relation between the limit fields (case $r_\varepsilon = k\varepsilon$)

In this section we still assume $r_\varepsilon = k\varepsilon$ and we derive a few relations between $\mathcal{U}^0$, $\mathcal{R}^0$, $\overline{u}^0$ on one hand, and $X$, $\Sigma$ on the other hand.

First, consider (4.53) which implies

$$\varepsilon\left(\frac{\partial \mathcal{U}_1^\varepsilon}{\partial x_3} - \mathcal{R}_2^\varepsilon\right) \to 0 \text{ strongly in } L^2(\Omega^+),$$

as $\varepsilon$ tends to 0. Then, (5.16) and (5.18) give

$$\frac{\partial \mathcal{U}_1^0}{\partial x_3} = \mathcal{R}_2^0 \text{ in } \Omega^+. \tag{5.23}$$

Indeed, using the second component in (4.53) leads to

$$\frac{\partial \mathcal{U}_2^0}{\partial x_3} = -\mathcal{R}_1^0 \text{ in } \Omega^+. \tag{5.24}$$

It follows that $\mathcal{U}_\alpha^0 \in L^2(\omega, H^2(]0,L[))$, for $\alpha = 1,2$.

Now, consider (4.38) which can be written, for any $(p,q) \in \mathcal{N}^\varepsilon$, as

$$u_1^\varepsilon(x_1,x_2,x_3) = \mathcal{U}^\varepsilon_{1|_{\Omega_\varepsilon^+}}(x_1,x_2,x_3) - \mathcal{R}^\varepsilon_{3|_{\Omega_\varepsilon^+}}(x_1,x_2,x_3)(x_2 - \varepsilon q) + \overline{u}_1^\varepsilon(x_1,x_2,x_3),$$

$$\text{if } (x_1,x_2) \in \mathcal{D}_{pq}^\varepsilon, \quad x_3 \in ]0,L[.$$

Then, for any $(p,q) \in \mathcal{N}^\varepsilon$,

$$\mathcal{T}^\varepsilon(u_1^\varepsilon)(x_1,x_2,x_3,X_1,X_2) = \mathcal{T}^\varepsilon\left(\mathcal{U}^\varepsilon_{1|_{\Omega_\varepsilon^+}}\right)(x_1,x_2,x_3,X_1,X_2) -$$

$$\mathcal{T}^\varepsilon\left(\mathcal{R}^\varepsilon_{3|_{\Omega_\varepsilon^+}}(x_2-\varepsilon q)\right)(x_1,x_2,x_3,X_1,X_2) + \mathcal{T}^\varepsilon(\overline{u}_1^\varepsilon)(x_1,x_2,x_3,X_1,X_2), \tag{5.25}$$

$$\text{if } (x_1,x_2) \in \left]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\right[ \times \left]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\right[, \quad x_3 \in ]0,L[, \quad (X_1,X_2) \in D.$$

Now remark that the function $\mathcal{U}^\varepsilon_{1|_{\Omega_\varepsilon^+}}(x_1,x_2,x_3)$ is constant on each $\mathcal{D}^\varepsilon_{pq}$, for almost any fixed $x_3$. As a consequence, the definition (5.1) of $\mathcal{T}^\varepsilon$ gives, for any $(p,q) \in \mathcal{N}^\varepsilon$,

$$\begin{aligned} &\mathcal{T}^\varepsilon(\mathcal{U}^\varepsilon_{1|_{\Omega_\varepsilon^+}}) = \mathcal{U}_1^\varepsilon, \\ &\text{if } (x_1,x_2) \in \left]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\right[ \times \left]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\right[, \quad x_3 \in ]0,L[, \quad (X_1,X_2) \in D. \end{aligned} \tag{5.26}$$

Since, for any $(p,q) \in \mathcal{N}^\varepsilon$,

$$\mathcal{T}^\varepsilon\left(\mathcal{R}^\varepsilon_{3|_{\Omega_\varepsilon^+}}(x_2-\varepsilon q)\right)(x_1,x_2,x_3,X_1,X_2) = r_\varepsilon X_2 \mathcal{R}_3^\varepsilon(x_1,x_2,x_3),$$

$$\text{if } (x_1,x_2) \in \left]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\right[ \times \left]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\right[, \quad x_3 \in ]0,L[, \quad (X_1,X_2) \in D,$$

and equality (5.25) leads to

$$\begin{aligned} &\mathcal{T}^\varepsilon(u_1^\varepsilon)(x_1,x_2,x_3,X_1,X_2) = \mathcal{U}_1^\varepsilon(x_1,x_2,x_3) - \\ &r_\varepsilon X_2 \mathcal{R}_3^\varepsilon(x_1,x_2,x_3) + \mathcal{T}^\varepsilon(\overline{u}_1^\varepsilon)(x_1,x_2,x_3,X_1,X_2) \quad \text{a.e. in } \Omega^+ \times D. \end{aligned} \tag{5.27}$$

In (5.27) we also have used the fact that

$$\mathcal{T}^\varepsilon(u_1^\varepsilon) = \mathcal{U}_1^\varepsilon = \mathcal{R}_3^\varepsilon = \mathcal{T}^\varepsilon(\overline{u}_1^\varepsilon) = 0,$$

$$\text{if } (x_1,x_2,x_3) \in \Omega^+ \setminus (\widetilde{\omega}_\varepsilon \times ]0,L[)\,.$$

In view of (5.13), (5.15), (5.16) and (5.18), by passing to the limit in (5.27), as $\varepsilon$ tends to zero, we obtain, since $r_\varepsilon = k\varepsilon$,

$$u_1^0(x_1,x_2,x_3,X_1,X_2) = \mathcal{U}_1^0(x_1,x_2,x_3).$$

Repeating the above arguments for $u_2^\varepsilon$, we conclude that,

$$\begin{aligned} &u_\alpha^0(x_1,x_2,x_3,X_1,X_2) = \mathcal{U}_\alpha^0(x_1,x_2,x_3), \\ &\qquad \text{for almost any } (x_1,x_2,x_3) \in \Omega^+, \quad (X_1,X_2) \in D, \quad \text{for } \alpha = 1,2. \end{aligned} \tag{5.28}$$

Remark that $u^0_\alpha$, for $\alpha = 1, 2$, do not depend on the variables $(X_1, X_2)$.

As far as $u^\varepsilon_3$ is concerned, we have by (4.41) for any $(p, q) \in \mathcal{N}^\varepsilon$,

$$\begin{aligned} &u^\varepsilon_3(x_1, x_2, x_3) = \mathcal{U}^\varepsilon_{3|_{\Omega^+_\varepsilon}}(x_1, x_2, x_3) + \mathcal{R}^\varepsilon_{1|_{\Omega^+_\varepsilon}}(x_1, x_2, x_3)(x_2 - \varepsilon q) - \\ &\mathcal{R}^\varepsilon_{2|_{\Omega^+_\varepsilon}}(x_1, x_2, x_3)(x_1 - \varepsilon p) + \overline{u}^\varepsilon_3(x_1, x_2, x_3), \qquad \text{if } (x_1, x_2) \in \mathcal{D}^\varepsilon_{pq}, \quad x_3 \in ]0, L[. \end{aligned} \tag{5.29}$$

First we have

$$\mathcal{T}^\varepsilon(\overline{u}^\varepsilon_3) \to 0 \text{ strongly in } L^2(\Omega^+ \times D), \tag{5.30}$$

because of (5.15).

Then, as above for $\mathcal{U}^\varepsilon_\alpha$ and $\mathcal{R}^\varepsilon_\alpha$, $\alpha = 1, 2$, for any $(p, q) \in \mathcal{N}^\varepsilon$, it results

$$\begin{cases} \mathcal{T}^\varepsilon(\mathcal{U}^\varepsilon_{3|_{\Omega^+_\varepsilon}}) = \mathcal{U}^\varepsilon_3, \\ \mathcal{T}^\varepsilon\left(\mathcal{R}^\varepsilon_{1|_{\Omega^+_\varepsilon}}(x_2 - \varepsilon q)\right)(x_1, x_2, x_3, X_1, X_2) = r_\varepsilon X_2 \mathcal{R}^\varepsilon_1(x_1, x_2, x_3), \\ \mathcal{T}^\varepsilon\left(\mathcal{R}^\varepsilon_{2|_{\Omega^+_\varepsilon}}(x_1 - \varepsilon p)\right)(x_1, x_2, x_3, X_1, X_2) = r_\varepsilon X_1 \mathcal{R}^\varepsilon_2(x_1, x_2, x_3), \\ \text{if } (x_1, x_2) \in \left]\varepsilon p - \dfrac{\varepsilon}{2}, \varepsilon p + \dfrac{\varepsilon}{2}\right[ \times \left]\varepsilon q - \dfrac{\varepsilon}{2}, \varepsilon q + \dfrac{\varepsilon}{2}\right[, \quad x_3 \in ]0, L[, \quad (X_1, X_2) \in D. \end{cases}$$

Proceeding as for $\mathcal{U}^\varepsilon_\alpha$ above, and using now (5.14), (5.17), (5.18) and (5.30), equality (5.29) implies that, since $r_\varepsilon = k\varepsilon$,

$$\begin{aligned} u^0_3(x_1, x_2, x_3, X_1, X_2) = \mathcal{U}^0_3(x_1, x_2, x_3) + kX_2\mathcal{R}^0_1(x_1, x_2, x_3) - kX_1\mathcal{R}^0_2(x_1, x_2, x_3), \\ \text{for almost any } (x_1, x_2, x_3) \in \Omega^+, \quad (X_1, X_2) \in D. \end{aligned} \tag{5.31}$$

Remark that, due to (5.23) and (5.24), relation (5.31) can be equivalently rewritten as

$$\begin{aligned} u^0_3(x_1, x_2, x_3, X_1, X_2) = \mathcal{U}^0_3(x_1, x_2, x_3) - kX_1\frac{\partial \mathcal{U}^0_1}{\partial x_3}(x_1, x_2, x_3) - kX_2\frac{\partial \mathcal{U}^0_2}{\partial x_3}(x_1, x_2, x_3), \\ \text{for almost any } (x_1, x_2, x_3) \in \Omega^+, \quad (X_1, X_2) \in D. \end{aligned} \tag{5.32}$$

We now turn to the identification of $X_{ij}$ (see (5.20)). In view of the decomposition of $u^\varepsilon$ given in (4.38) and (4.39), we have

$$\gamma_{\alpha\beta}(u^\varepsilon) = \gamma_{\alpha\beta}(\overline{u}^\varepsilon) \text{ a.e. in } \Omega^+_\varepsilon, \text{ for } \alpha, \beta = 1, 2. \tag{5.33}$$

Appealing now to the rule for the derivation of an unfold field given in (e) of Lemma 5.1, we obtain

$$r_\varepsilon \mathcal{T}^\varepsilon\left(\gamma_{\alpha\beta}(u^\varepsilon)\right) = \Gamma_{\alpha\beta}\left(\mathcal{T}^\varepsilon(\overline{u}^\varepsilon)\right) \text{ a.e. in } \Omega^+ \times D, \text{ for } \alpha, \beta = 1, 2, \tag{5.34}$$

where for any field $v$, say in $\left(L^2(\Omega^+; H^1(D))\right)^3$, we have set

$$\Gamma_{\alpha\beta}(v) = \frac{1}{2}\left(\partial_{X_\beta} v_\alpha + \partial_{X_\alpha} v_\beta\right) \text{ a.e. in } \Omega^+ \times D, \text{ for } \alpha, \beta = 1, 2. \tag{5.35}$$

Dividing (5.34) by $\varepsilon$ and passing to the limit, as $\varepsilon$ tends to zero, yields using (5.15) and (5.20)

$$kX_{\alpha\beta} = \Gamma_{\alpha\beta}(\overline{u}^0) \text{ a.e. in } \Omega^+ \times D, \text{ for } \alpha, \beta = 1, 2. \tag{5.36}$$

Let us now consider $\gamma_{13}(u^\varepsilon)$. Fix $(p, q) \in \mathcal{N}^\varepsilon$. In view of (4.38) and (4.41), we have

$$\begin{aligned}
&\gamma_{13}(u^\varepsilon)(x_1, x_2, x_3) = \\
&\frac{1}{2}\left[\frac{\partial \mathcal{U}_1^\varepsilon}{\partial x_3}(x_1, x_2, x_3) - \frac{\partial \mathcal{R}_3^\varepsilon}{\partial x_3}(x_1, x_2, x_3)(x_2 - \varepsilon q) + \frac{\partial \overline{u}_1^\varepsilon}{\partial x_3}(x_1, x_2, x_3) - \right. \\
&\left. \mathcal{R}_2^\varepsilon(x_1, x_2, x_3) + \frac{\partial \overline{u}_3^\varepsilon}{\partial x_1}(x_1, x_2, x_3)\right], \text{ if } (x_1, x_2) \in \mathcal{D}_{pq}^\varepsilon, \ x_3 \in ]0, L[.
\end{aligned} \tag{5.37}$$

We apply the unfolding operator to both hand of (5.37) and consider the behavior of each term appearing in the right hand side. Since again $\mathcal{U}_i^\varepsilon$ and $\mathcal{R}_i^\varepsilon$ are constant on each $\mathcal{D}_{pq}^\varepsilon$, we have for $(p, q) \in \mathcal{N}^\varepsilon$ (as for (5.26)),

$$\mathcal{T}^\varepsilon\left(\frac{\partial \mathcal{U}_1^\varepsilon}{\partial x_3}_{|_{\Omega_\varepsilon^+}} - \mathcal{R}^\varepsilon_{2|_{\Omega_\varepsilon^+}}\right) = \frac{\partial \mathcal{U}_1^\varepsilon}{\partial x_3} - \mathcal{R}_2^\varepsilon, \tag{5.38}$$

$$\mathcal{T}^\varepsilon\left(\frac{\partial \mathcal{R}_3^\varepsilon}{\partial x_3}_{|_{\Omega_\varepsilon^+}}(x_2 - \varepsilon q)\right) = r_\varepsilon X_2 \frac{\partial \mathcal{R}_3^\varepsilon}{\partial x_3}, \tag{5.39}$$

$$\mathcal{T}^\varepsilon\left(\mathcal{R}^\varepsilon_{2|_{\Omega_\varepsilon^+}}\right) = \mathcal{R}_2^\varepsilon, \tag{5.40}$$

if $(x_1, x_2) \in \left]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\right[ \times \left]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\right[$, $\quad x_3 \in ]0, L[$, $\quad (X_1, X_2) \in D$. Using the rules (e) of Lemma 5.1 for the derivations of an unfold field, yields

$$r_\varepsilon \mathcal{T}^\varepsilon\left(\frac{\partial \overline{u}_3^\varepsilon}{\partial x_1}\right) = \frac{\partial(\mathcal{T}^\varepsilon(\overline{u}_3^\varepsilon))}{\partial X_1} \text{ a.e. in } \Omega^+ \times D, \tag{5.41}$$

$$\mathcal{T}^\varepsilon\left(\frac{\partial \overline{u}_1^\varepsilon}{\partial x_3}\right) = \frac{\partial(\mathcal{T}^\varepsilon(\overline{u}_1^\varepsilon))}{\partial x_3} \text{ a.e. in } \Omega^+ \times D. \tag{5.42}$$

Then (5.37)÷(5.42) give

$$\begin{aligned}
&\mathcal{T}^\varepsilon\left(\gamma_{13}(u^\varepsilon)\right) = \\
&\frac{1}{2}\left[\left(\frac{\partial \mathcal{U}_1^\varepsilon}{\partial x_3} - \mathcal{R}_2^\varepsilon\right) - r_\varepsilon X_2 \frac{\partial \mathcal{R}_3^\varepsilon}{\partial x_3} + \frac{\partial(\mathcal{T}^\varepsilon(\overline{u}_1^\varepsilon))}{\partial x_3} + \frac{1}{r_\varepsilon}\frac{\partial(\mathcal{T}^\varepsilon(\overline{u}_3^\varepsilon))}{\partial X_1}\right] \text{ a.e. in } \Omega^+ \times D.
\end{aligned} \tag{5.43}$$

Convergences (5.15), (5.18), (5.19) and (5.20) allow to pass to the limit in (5.43), and to obtain

$$X_{13} = \frac{1}{2}\left[Z_1 - X_2 k \frac{\partial \mathcal{R}_3^0}{\partial x_3} + \frac{1}{k}\frac{\partial \overline{u}_3^0}{\partial X_1}\right] \text{ a.e. in } \Omega^+ \times D,$$

which can be written as

$$X_{13} = \frac{1}{2}\left[\frac{\partial}{\partial X_1}\left(X_1 Z_1 + \frac{1}{k}\overline{u}_3^0\right) - X_2 k \frac{\partial \mathcal{R}_3^0}{\partial x_3}\right] \text{ a.e. in } \Omega^+ \times D. \tag{5.44}$$

Proceeding as above to identify $X_{13}$, we obtain

$$X_{23} = \frac{1}{2}\left[\frac{\partial}{\partial X_2}\left(X_2 Z_2 + \frac{1}{k}\overline{u}_3^0\right) + X_1 k \frac{\partial \mathcal{R}_3^0}{\partial x_3}\right] \text{ a.e. in } \Omega^+ \times D. \tag{5.45}$$

To derive $X_{33}$, we write, for any $(p,q) \in \mathcal{N}^\varepsilon$, in view of (4.41),

$$\begin{aligned}&\gamma_{33}(u^\varepsilon)(x_1,x_2,x_3) = \\ &\frac{\partial \mathcal{U}_3^\varepsilon}{\partial x_3}(x_1,x_2,x_3) + \frac{\partial \overline{u}_3^\varepsilon}{\partial x_3}(x_1,x_2,x_3) + \frac{\partial \mathcal{R}_1^\varepsilon}{\partial x_3}(x_1,x_2,x_3)(x_2 - \varepsilon q) - \\ &\frac{\partial \mathcal{R}_2^\varepsilon}{\partial x_3}(x_1,x_2,x_3)(x_1 - \varepsilon p) \text{ if } (x_1,x_2) \in \mathcal{D}_{pq}^\varepsilon,\ \ x_3 \in ]0,L[.\end{aligned} \tag{5.46}$$

The same type of calculations that leads to the expression of $X_{13}$, which is not repeated here, gives

$$X_{33} = \frac{\partial \mathcal{U}_3^0}{\partial x_3} + k X_2 \frac{\partial \mathcal{R}_1^0}{\partial x_3} - k X_1 \frac{\partial \mathcal{R}_2^0}{\partial x_3} \text{ a.e. in } \Omega^+ \times D. \tag{5.47}$$

According to (5.23) and (5.24), $X_{33}$ can be expressed as

$$X_{33} = \frac{\partial \mathcal{U}_3^0}{\partial x_3} - k X_1 \frac{\partial^2 \mathcal{U}_1^0}{\partial x_3^2} - k X_2 \frac{\partial^2 \mathcal{U}_2^0}{\partial x_3^2} \text{ a.e. in } \Omega^+ \times D. \tag{5.48}$$

To conclude this subsection, we deduce from the constitutive law (2.12), from (5.20) and (5.21) and from the above expression of $X_{ij}$ that

$$\begin{aligned}\Sigma_{11} = \frac{1}{k}\left[(\lambda+2\mu)\Gamma_{11}(\overline{u}^0) + \lambda\Gamma_{22}(\overline{u}^0)\right] + \lambda\left(\frac{\partial \mathcal{U}_3^0}{\partial x_3} - kX_1\frac{\partial^2 \mathcal{U}_1^0}{\partial x_3^2} - kX_2\frac{\partial^2 \mathcal{U}_2^0}{\partial x_3^2}\right)\\ \text{a.e. in } \Omega^+ \times D,\end{aligned} \tag{5.49}$$

$$\begin{aligned}\Sigma_{22} = \frac{1}{k}\left[(\lambda+2\mu)\Gamma_{22}(\overline{u}^0) + \lambda\Gamma_{11}(\overline{u}^0)\right] + \lambda\left(\frac{\partial \mathcal{U}_3^0}{\partial x_3} - kX_1\frac{\partial^2 \mathcal{U}_1^0}{\partial x_3^2} - kX_2\frac{\partial^2 \mathcal{U}_2^0}{\partial x_3^2}\right)\\ \text{a.e. in } \Omega^+ \times D,\end{aligned} \tag{5.50}$$

$$\Sigma_{12} = 2\frac{\mu}{k}\Gamma_{12}(\overline{u}^0) \text{ a.e. in } \Omega^+ \times D, \tag{5.51}$$

$$\Sigma_{13} = \mu \left[ \frac{\partial}{\partial X_1} \left( X_1 Z_1 + \frac{1}{k} \overline{u}_3^0 \right) - k X_2 \frac{\partial \mathcal{R}_3^0}{\partial x_3} \right] \text{ a.e. in } \Omega^+ \times D, \tag{5.52}$$

$$\Sigma_{23} = \mu \left[ \frac{\partial}{\partial X_2} \left( X_2 Z_2 + \frac{1}{k} \overline{u}_3^0 \right) + k X_1 \frac{\partial \mathcal{R}_3^0}{\partial x_3} \right] \text{ a.e. in } \Omega^+ \times D, \tag{5.53}$$

$$\Sigma_{33} = (\lambda + 2\mu) \left( \frac{\partial \mathcal{U}_3^0}{\partial x_3} - k X_1 \frac{\partial^2 \mathcal{U}_1^0}{\partial x_3^2} - k X_2 \frac{\partial^2 \mathcal{U}_2^0}{\partial x_3^2} \right) + \frac{\lambda}{k} \left( \Gamma_{11}(\overline{u}^0) + \Gamma_{22}(\overline{u}^0) \right) \quad \text{a.e. in } \Omega^+ \times D. \tag{5.54}$$

## 5.5 Limit kinematic conditions (case $r_\varepsilon = k\varepsilon$)

In this section we derive, in the case $r_\varepsilon = k\varepsilon$, the kinematic conditions on the "type" displacement fields $\mathcal{U}_i^0$, $\mathcal{R}_i^0$ and $\overline{u}_i^0$. In particular, we derive the kinematic junction conditions between the "continuum" of rods in $\Omega^+$ and the $3d$ body in $\Omega^-$.

First of all, comparing (4.28), (5.16) on the one hand, and (4.25), (5.18) on the other hand leads to to

$$\mathcal{U}_\alpha^0(x_1, x_2, 0) = 0 \text{ a.e. in } \omega, \text{ for } \alpha = 1, 2, \tag{5.55}$$

and

$$\mathcal{R}_i^0(x_1, x_2, 0) = 0 \text{ a.e. in } \omega, \text{ for } i = 1, 2, 3. \tag{5.56}$$

This last relation together with (5.23), (5.24) gives

$$\frac{\partial \mathcal{U}_\alpha^0}{\partial x_3}(x_1, x_2, 0) = 0 \text{ a.e. in } \omega, \text{ for } \alpha = 1, 2. \tag{5.57}$$

We now turn to the transmission condition between $\mathcal{U}_3^0$ and $u_3^-$ on $\omega$.

Since $u^\varepsilon \in H^1(\Omega_\varepsilon)$, recalling Remark 5.2, one can define $\mathcal{T}^\varepsilon(u_3^\varepsilon)$ on $]-l, L[\times\omega\times D$ (still by (5.1)). One has $\dfrac{\partial(\mathcal{T}^\varepsilon(u_3^\varepsilon))}{\partial x_3} = \mathcal{T}^\varepsilon\left(\dfrac{\partial u_3^\varepsilon}{\partial x_3}\right)$, and then the weak convergences (5.14) and (5.20) imply that $\mathcal{T}^\varepsilon(u_3^\varepsilon)$ is bounded in $L^2(\omega \times D, H^1(]-l, L[))$. Then, $\mathcal{T}^\varepsilon(u_3^\varepsilon) \rightharpoonup u_3^*$ weakly in $L^2(\omega \times D, H^1(]-l, L[)) = H^1(]-l, L[, L^2(\omega \times D))$ (at least for a subsequence). Due to (5.14) and (5.31), we first have

$$u_3^* = \mathcal{U}_3^0 + k X_2 \mathcal{R}_1^0 - k X_1 \mathcal{R}_2^0 \text{ in } \Omega^+ \times D.$$

Now, from (5.22), $u_3^\varepsilon \to u_3^-$ strongly in $L^2(\Omega^-)$, and using again Remark 5.2, we know that $\mathcal{T}^\varepsilon(u_3^\varepsilon) \to u_3^-$ strongly in $L^2(\Omega^- \times D)$, so that $u_3^* = u_3^-$ in $\Omega^- \times D$. Since $u_3^* \in C^0(]-l, L[, L^2(\omega \times D))$, we obtain

$$u_3^-(x_1, x_2, 0) = \mathcal{U}_3^0(x_1, x_2, 0) + k X_2 \mathcal{R}_1^0(x_1, x_2, 0) - k X_1 \mathcal{R}_2^0(x_1, x_2, 0) \quad \text{a.e. in } \omega \times D. \tag{5.58}$$

This last relation together with (5.56) (actually it gives again (5.56) because $(X_1, X_2)$ are arbitrary in $D$) leads to

$$u_3^-(x_1, x_2, 0) = \mathcal{U}_3^0(x_1, x_2, 0) \text{ a.e. in } \omega \times D, \tag{5.59}$$

which is the transmission condition on the vertical displacement of the rods and the plate.

To end this section, we derive the kinematic conditions on $\overline{u}^0$ which follow from (3.8)÷(3.10). Recall that by definition (4.12) of $\overline{u}^\varepsilon$ and (3.8)÷(3.10), we have for any for any $(p,q)\in\mathcal{N}^\varepsilon$

$$\int_{\mathcal{D}^\varepsilon_{pq}} \overline{u}^\varepsilon_i(x_1,x_2,x_3)dx_1dx_2 = 0 \quad \text{ for } i=1,2,3, \tag{5.60}$$

$$\int_{\mathcal{D}^\varepsilon_{pq}} (x_1-\varepsilon p)\overline{u}^\varepsilon_3(x_1,x_2,x_3)dx_1dx_2 = \int_{\mathcal{D}^\varepsilon_{pq}} (x_2-\varepsilon q)\overline{u}^\varepsilon_3(x_1,x_2,x_3)dx_1dx_2 = 0, \tag{5.61}$$

$$\int_{\mathcal{D}^\varepsilon_{pq}} \big[(x_1-\varepsilon p)\overline{u}^\varepsilon_2(x_1,x_2,x_3) - (x_2-\varepsilon q)\overline{u}^\varepsilon_1(x_1,x_2,x_3)\big]dx_1dx_2 = 0, \tag{5.62}$$

for almost any $x_3$ in $]0,L[$.

Let $\varphi$ be a function of $C^\infty_0(\Omega^+)$. For $\varepsilon$ small enough the support of $\varphi$ is included in $\widetilde{\omega}_\varepsilon\times]0,L[$. Then, define $\widetilde{\varphi}$ in $\Omega^+$ as follows: for any $(p,q)\in\mathcal{N}^\varepsilon$, $\widetilde{\varphi}_\varepsilon(x_1,x_2,x_3)=\varphi(\varepsilon p,\varepsilon q,x_3)$, if $(x_1,x_2)\in\Big]\varepsilon p-\dfrac{\varepsilon}{2},\varepsilon p+\dfrac{\varepsilon}{2}\Big[\times\Big]\varepsilon q-\dfrac{\varepsilon}{2},\varepsilon q+\dfrac{\varepsilon}{2}\Big[$ and $x_3\in]0,L[$, $\widetilde{\varphi}_\varepsilon(x_1,x_2,x_3)=0$ otherwise.

Due to (5.60)÷(5.62), it follows that

$$\int_{\Omega^+_\varepsilon} \widetilde{\varphi}_\varepsilon \overline{u}^\varepsilon_i dx_1dx_2dx_3 = 0, \quad \text{ for } i=1,2,3, \tag{5.63}$$

$$\begin{cases} \displaystyle\sum_{(p,q)\in\mathcal{N}^\varepsilon}\int_0^L\int_{\mathcal{D}^\varepsilon_{pq}} \widetilde{\varphi}_\varepsilon(x_1-\varepsilon p)\overline{u}^\varepsilon_3 dx_1dx_2dx_3 = 0,\\[2ex] \displaystyle\sum_{(p,q)\in\mathcal{N}^\varepsilon}\int_0^L\int_{\mathcal{D}^\varepsilon_{pq}} \widetilde{\varphi}_\varepsilon(x_2-\varepsilon q)\overline{u}^\varepsilon_3 dx_1dx_2dx_3 = 0, \end{cases} \tag{5.64}$$

$$\sum_{(p,q)\in\mathcal{N}^\varepsilon}\int_0^L\int_{\mathcal{D}^\varepsilon_{pq}} \widetilde{\varphi}_\varepsilon\left[(x_1-\varepsilon p)\overline{u}^\varepsilon_2-(x_2-\varepsilon q)\overline{u}^\varepsilon_1\right]dx_1dx_2dx_3 = 0. \tag{5.65}$$

In term of the unfolding operator $\mathcal{T}^\varepsilon$, (5.63) reads as

$$\int_{\Omega^+\times D} \mathcal{T}^\varepsilon\left(\widetilde{\varphi}_\varepsilon\right)\mathcal{T}^\varepsilon\left(\overline{u}^\varepsilon\right)dx_1dx_2dx_3dX_1dX_2 = 0. \tag{5.66}$$

Since $\widetilde{\varphi}_\varepsilon$ is constant in each $\mathcal{D}^\varepsilon_{pq}$ for fixed $x_3$, $\mathcal{T}^\varepsilon\left(\widetilde{\varphi}_\varepsilon\right)=\widetilde{\varphi}_\varepsilon$. Indeed $\widetilde{\varphi}_\varepsilon\to\varphi$ strongly in $L^2(\Omega^+)$, so that (5.15) implies that

$$\int_{\Omega^+\times D} \varphi(x_1,x_2,x_3)\overline{u}^0(x_1,x_2,x_3,X_1,X_2)dx_1dx_2dx_3dX_1dX_2 = 0,$$

from which we deduce that for almost any $(x_1,x_2,x_3)\in\Omega^+$

$$\int_D \overline{u}^0(x_1,x_2,x_3,X_1,X_2)dX_1dX_2 = 0. \tag{5.67}$$

The same technique permits to obtain from (5.64) and (5.65) that for almost any $(x_1, x_2, x_3) \in \Omega^+$

$$\int_D X_\alpha \overline{u}_3^0(x_1, x_2, x_3, X_1, X_2) dX_1 dX_2 = 0, \qquad \text{for } \alpha = 1, 2, \tag{5.68}$$

$$\int_D \big[X_1 \overline{u}_2^0(x_1, x_2, x_3, X_1, X_2) - X_2 \overline{u}_1^0(x_1, x_2, x_3, X_1, X_2)\big] dX_1 dX_2 = 0. \tag{5.69}$$

# 6 The limit problem (case $r_\varepsilon = k\varepsilon$)

In this section we derive the equations satisfied by $\mathcal{U}^0$, $\mathcal{R}^0$, $\overline{u}^0$ and $u^-$.

As a starting point, and in order to pass to the limit as the parameter $\varepsilon$ tends to zero, we write (2.18) in terms of the unfolding operator $\mathcal{T}^\varepsilon$ in $\Omega_\varepsilon^+$. It gives, recalling $r_\varepsilon = k\varepsilon$ and (a) of Lemma 5.1,

$$\begin{aligned} & k^2 \sum_{i,j=1}^{3} \int_{\Omega^+ \times D} \mathcal{T}^\varepsilon\left(\sigma_{ij}^\varepsilon\right) \mathcal{T}^\varepsilon\left(\gamma_{ij}(v)\right) dx_1 dx_2 dx_3 dX_1 dX_2 + \\ & \sum_{i,j=1}^{3} \int_{\Omega^-} \sigma_{ij}^\varepsilon \gamma_{ij}(v) dx_1 dx_2 dx_3 = k^2 \sum_{i=1}^{3} \int_{\Omega^+ \times D} \mathcal{T}^\varepsilon\left(f_i^\varepsilon\right) \mathcal{T}^\varepsilon\left(v_i\right) dx_1 dx_2 dx_3 dX_1 dX_2 \\ & + \sum_{i=1}^{3} \int_{\Omega^-} f_i^\varepsilon v_i dx_1 dx_2 dx_3, \quad \forall v \in V_\varepsilon. \end{aligned} \tag{6.1}$$

We will pass to the limit in (6.1) when $\varepsilon$ tends to zero, and the advantage in introducing $\mathcal{T}^\varepsilon$ is that now the domain $\Omega^+ \times D$ is fixed. The limit process is achieved with specific choices of the test function $v$.

The section is organized as follows. First, we obtain the relations between $\overline{u}_\alpha^0$ and $\mathcal{U}_3^0$ and we show that $\overline{u}_3^0 = 0$. Then, we obtain the system of partial differential equations verified by $\mathcal{U}^0$ and $u^-$. At least, we prove strong convergence of the energy.

## 6.1 Equations for $\overline{u}$ (case $r_\varepsilon = k\varepsilon$)

Let $\varphi$ be in $C_0^\infty(\omega)$ and $\overline{v}$ be a function of $\left(C^\infty(\overline{D} \times [0, L])\right)^3$ such that $\overline{v}(X_1, X_2, 0) = 0$. In (6.1), we choose the function $v^\varepsilon$ defined for $(x_1, x_2, x_3) \in \Omega_\varepsilon^+$ by

$$\begin{gathered} v^\varepsilon(x_1, x_2, x_3) = r_\varepsilon \varphi(\varepsilon p, \varepsilon q) \overline{v}\left(\frac{x_1 - \varepsilon p}{r_\varepsilon}, \frac{x_2 - \varepsilon q}{r_\varepsilon}, x_3\right) \\ \text{if } (x_1, x_2) \in \mathcal{D}_{pq}^\varepsilon,\ x_3 \in ]0, L[, \text{ for } (p, q) \in \mathcal{N}^\varepsilon, \end{gathered} \tag{6.2}$$

and

$$v^\varepsilon(x_1, x_2, x_3) = 0 \text{ if } (x_1, x_2, x_3) \in \Omega^-. \tag{6.3}$$

Then $v^\varepsilon \in \left(C^\infty(\overline{\Omega_\varepsilon^+})\right)^3 \cap V_\varepsilon$.

In $\Omega_\varepsilon^+$ we have

$$\gamma_{11}(v^\varepsilon)(x_1,x_2,x_3)=\varphi(\varepsilon p,\varepsilon q)\frac{\partial\overline{v}_1}{\partial X_1}\left(\frac{x_1-\varepsilon p}{r_\varepsilon},\frac{x_2-\varepsilon q}{r_\varepsilon},x_3\right),$$

$$\gamma_{22}(v^\varepsilon)(x_1,x_2,x_3)=\varphi(\varepsilon p,\varepsilon q)\frac{\partial\overline{v}_2}{\partial X_2}\left(\frac{x_1-\varepsilon p}{r_\varepsilon},\frac{x_2-\varepsilon q}{r_\varepsilon},x_3\right),$$

$$\gamma_{12}(v^\varepsilon)(x_1,x_2,x_3)=\frac{\varphi(\varepsilon p,\varepsilon q)}{2}\left[\frac{\partial\overline{v}_1}{\partial X_2}+\frac{\partial\overline{v}_2}{\partial X_1}\right]\left(\frac{x_1-\varepsilon p}{r_\varepsilon},\frac{x_2-\varepsilon q}{r_\varepsilon},x_3\right),$$

$$\gamma_{13}(v^\varepsilon)(x_1,x_2,x_3)=\frac{\varphi(\varepsilon p,\varepsilon q)}{2}\left[r_\varepsilon\frac{\partial\overline{v}_1}{\partial x_3}+\frac{\partial\overline{v}_3}{\partial X_1}\right]\left(\frac{x_1-\varepsilon p}{r_\varepsilon},\frac{x_2-\varepsilon q}{r_\varepsilon},x_3\right),$$

$$\gamma_{23}(v^\varepsilon)(x_1,x_2,x_3)=\frac{\varphi(\varepsilon p,\varepsilon q)}{2}\left[r_\varepsilon\frac{\partial\overline{v}_2}{\partial x_3}+\frac{\partial\overline{v}_3}{\partial X_2}\right]\left(\frac{x_1-\varepsilon p}{r_\varepsilon},\frac{x_2-\varepsilon q}{r_\varepsilon},x_3\right),$$

$$\gamma_{33}(v^\varepsilon)(x_1,x_2,x_3)=\varphi(\varepsilon p,\varepsilon q)r_\varepsilon\frac{\partial\overline{v}_3}{\partial x_3}\left(\frac{x_1-\varepsilon p}{r_\varepsilon},\frac{x_2-\varepsilon q}{r_\varepsilon},x_3\right),$$

if $(x_1,x_2)\in\mathcal{D}^\varepsilon_{pq}$, $x_3\in]0,L[$, for $(p,q)\in\mathcal{N}^\varepsilon$.

Define the function $\widetilde{\varphi}^\varepsilon$ in $\omega$ by

$$\widetilde{\varphi}^\varepsilon(x_1,x_2)=\begin{cases}\varphi(\varepsilon p,\varepsilon q),\ \text{if } (x_1,x_2)\in\left]\varepsilon p-\frac{\varepsilon}{2},\varepsilon p+\frac{\varepsilon}{2}\right[\times\left]\varepsilon q-\frac{\varepsilon}{2},\varepsilon q+\frac{\varepsilon}{2}\right[,\\ \\ 0,\ \text{if } (x_1,x_2)\in\omega\setminus\widetilde{\omega}_\varepsilon,\end{cases}\tag{6.4}$$

then applying the unfolding operator to $\gamma(v^\varepsilon)$ leads to

$$\mathcal{T}^\varepsilon\left(\gamma_{\alpha\beta}(v^\varepsilon)\right)=\widetilde{\varphi}^\varepsilon\Gamma_{\alpha\beta}(\overline{v})\ \text{a.e. in } \Omega^+\times D,\ \text{for } \alpha,\beta=1,2,\tag{6.5}$$

$$\mathcal{T}^\varepsilon\left(\gamma_{\alpha3}(v^\varepsilon)\right)=\widetilde{\varphi}^\varepsilon\frac{1}{2}\left[r_\varepsilon\frac{\partial\overline{v}_\alpha}{\partial x_3}+\frac{\partial\overline{v}_3}{\partial X_\alpha}\right]\ \text{a.e. in } \Omega^+\times D,\ \text{for } \alpha=1,2,\tag{6.6}$$

$$\mathcal{T}^\varepsilon\left(\gamma_{33}(v^\varepsilon)\right)=\widetilde{\varphi}^\varepsilon r_\varepsilon\Gamma_{33}(\overline{v})\ \text{a.e. in } \Omega^+\times D,\tag{6.7}$$

where $\Gamma_{ij}$ is defined in (5.35).

Since $\widetilde{\varphi}^\varepsilon\to\varphi$ strongly in $L^2(\omega)$ as $\varepsilon\to 0$, we obtain using the convergence (5.21)

$$\begin{aligned}&\lim_{\varepsilon\to 0}\sum_{i,j=1}^{3}\int_{\Omega^+\times D}\mathcal{T}^\varepsilon(\sigma^\varepsilon_{ij})\mathcal{T}^\varepsilon\left(\gamma_{ij}(v^\varepsilon)\right)dx_1dx_2dx_3dX_1dX_2=\\ &\sum_{\alpha,\beta=1}^{2}\int_{\Omega^+\times D}\varphi\Sigma_{\alpha\beta}\Gamma_{\alpha\beta}(\overline{v})dx_1dx_2dx_3dX_1dX_2+\\ &\sum_{\alpha=1}^{2}\int_{\Omega^+\times D}\varphi\Sigma_{\alpha3}\frac{\partial\overline{v}_3}{\partial X_\alpha}dx_1dx_2dx_3dX_1dX_2,\end{aligned}\tag{6.8}$$

because $r_\varepsilon = k\varepsilon \to 0$ as $\varepsilon \to 0$.

As far as the right hand side of (6.1) is concerned, we first have by assumption (2.19), (2.20) and (b) of Lemma 5.1

$$\mathcal{T}^\varepsilon(f^\varepsilon_\alpha) = r_\varepsilon \mathcal{T}^\varepsilon(f_\alpha) \to 0 \text{ strongly in } L^2(\Omega^+ \times D), \tag{6.9}$$

and

$$\mathcal{T}^\varepsilon(f^\varepsilon_3) = \mathcal{T}^\varepsilon(f_3) \to f^+_3 \text{ strongly in } L^2(\Omega^+ \times D). \tag{6.10}$$

Moreover, with (6.2),

$$\mathcal{T}^\varepsilon(v^\varepsilon) = \widetilde{\varphi}^\varepsilon r_\varepsilon \overline{v} \text{ a.e. in } \Omega^+ \times D. \tag{6.11}$$

then, we obtain from (6.1), (6.8) and (6.9)÷(6.11)

$$\begin{aligned}&\sum_{\alpha,\beta=1}^{2} \int_{\Omega^+\times D} \varphi \Sigma_{\alpha\beta} \Gamma_{\alpha\beta}(\overline{v}) dx_1 dx_2 dx_3 dX_1 dX_2 + \\ &\sum_{\alpha=1}^{2} \int_{\Omega^+\times D} \varphi \Sigma_{\alpha 3} \frac{\partial \overline{v}_3}{\partial X_\alpha} dx_1 dx_2 dx_3 dX_1 dX_2 = 0,\end{aligned} \tag{6.12}$$

and this equality holds true for any $\varphi \in C^\infty_0(\omega)$ and $\overline{v} \in C^\infty(\overline{D}\times[0,L])$ such that $\overline{v}(X_1,X_2,0) = 0$. Since $\varphi$ is arbitrary, (6.12) can be indeed localized a.e. in $\omega$.

We first choose $\overline{v}_1 = \overline{v}_2 = 0$ a.e. in $D\times]0,L[$. According to (5.52) and (5.54), it yields:

$$\begin{aligned}&\int_{D\times]0,L[} \left[ \frac{\partial}{\partial X_1}\left( X_1 Z_1 + \frac{1}{k}\overline{u}^0_3 \right) - k X_2 \frac{\partial \mathcal{R}^0_3}{\partial x_3} \right] \frac{\partial \overline{v}_3}{\partial X_1} dX_1 dX_2 dx_3 + \\ &\int_{D\times]0,L[} \left[ \frac{\partial}{\partial X_2}\left( X_2 Z_2 + \frac{1}{k}\overline{u}^0_3 \right) + k X_1 \frac{\partial \mathcal{R}^0_3}{\partial x_3} \right] \frac{\partial \overline{v}_3}{\partial X_2} dX_1 dX_2 dx_3 = 0 \text{ a.e. in } \omega.\end{aligned} \tag{6.13}$$

Remarking that (6.13) can be also localized with respect to $x_3$ and recalling that $Z_1$, $Z_2$ and $\mathcal{R}^0_3$ do not depend on $(X_1, X_2)$, it implies that the function $w = X_1 Z_1 + X_2 Z_2 + \dfrac{1}{k}\overline{u}^0_3$ satisfies

$$\begin{cases} -\dfrac{\partial^2 w}{\partial X_1^2} - \dfrac{\partial^2 w}{\partial X_2^2} = 0 \text{ in } D, \text{ a.e. in } \Omega^+, \\ \dfrac{\partial w}{\partial n} = 0 \text{ in } \partial D, \text{ a.e. in } \Omega^+, \end{cases}$$

because on $\partial D$, $X_1 n_1 - X_2 n_2 = 0$ a.e.. But by (5.67), $w$ also satisfies $\displaystyle\int_D w\, dX_1 dX_2 = 0$, for almost any $(x_1, x_2, x_3) \in \Omega^+$. As a consequence we deduce that $w = 0$, that is

$$\overline{u}^0_3 = -k(X_1 Z_1 + X_2 Z_2) \text{ a.e. in } \Omega^+ \times D.$$

At least, using the kinematic condition (5.68) on $\overline{u}_3^0$, we obtain $Z_1 = Z_2 = 0$ and

$$\overline{u}_3^0 = 0. \tag{6.14}$$

Remark that taking into account (6.14), the expressions (5.52) and (5.53) simplify to give

$$\Sigma_{13} = -\mu k X_2 \frac{\partial \mathcal{R}_3^0}{\partial x_3} \text{ a.e. in } \Omega^+ \times D, \tag{6.15}$$

$$\Sigma_{23} = \mu k X_1 \frac{\partial \mathcal{R}_3^0}{\partial x_3} \text{ a.e. in } \Omega^+ \times D. \tag{6.16}$$

Now we choose $\overline{v}_3 = 0$ in (6.12), using (5.49)÷(5.51), it leads to

$$\begin{aligned}
&\int_{D\times]0,L[} \frac{\lambda + 2\mu}{k} \left[\Gamma_{11}(\overline{u}^0)\Gamma_{11}(\overline{v}) + \Gamma_{22}(\overline{u}^0)\Gamma_{22}(\overline{v})\right] dX_1 dX_2 dx_3 + \\
&\int_{D\times]0,L[} \frac{\lambda}{k} \left[\Gamma_{11}(\overline{u}^0)\Gamma_{22}(\overline{v}) + \Gamma_{22}(\overline{u}^0)\Gamma_{11}(\overline{v})\right] dX_1 dX_2 dx_3 + \\
&\int_{D\times]0,L[} \frac{4\mu}{k} \Gamma_{12}(\overline{u}^0)\Gamma_{12}(\overline{v}) dX_1 dX_2 dx_3 = \\
&-\lambda \int_{D\times]0,L[} \left(\frac{\partial \mathcal{U}_3^0}{\partial x_3} - kX_1 \frac{\partial^2 \mathcal{U}_1^0}{\partial x_3^2} - kX_2 \frac{\partial^2 \mathcal{U}_2^0}{\partial x_3^2}\right) \left(\Gamma_{11}(\overline{v}) + \Gamma_{22}(\overline{v})\right) dX_1 dX_2 dx_3,
\end{aligned} \tag{6.17}$$

for any $\overline{v}_\alpha \in C^\infty(\overline{D}\times[0, L])$ such that $\overline{v}_\alpha(X_1, X_2, 0) = 0$ and then for any $\overline{v}_\alpha \in L^2(]0, L[; H^1(D))$, $\alpha = 1, 2$.

Actually, and after localization with respect to $x_3$, the variational problem (6.17) corresponds to classical $2d$ elastic problem for $(\overline{u}_1^0, \overline{u}_2^0)$ with constant forces on $D$ or on $\partial D$. Taking into account the kinematic conditions (5.60) and (5.62), the unique solution of (6.17) is given by

$$\overline{u}_1^0 = \nu \left\{ -kX_1 \frac{\partial \mathcal{U}_3^0}{\partial x_3} + k^2 \frac{X_1^2 - X_2^2}{2} \frac{\partial^2 \mathcal{U}_1^0}{\partial x_3^2} + k^2 X_1 X_2 \frac{\partial^2 \mathcal{U}_2^0}{\partial x_3^2} \right\}, \tag{6.18}$$

$$\overline{u}_2^0 = \nu \left\{ -kX_2 \frac{\partial \mathcal{U}_3^0}{\partial x_3} + k^2 X_1 X_2 \frac{\partial^2 \mathcal{U}_1^0}{\partial x_3^2} + k^2 \frac{X_2^2 - X_1^2}{2} \frac{\partial^2 \mathcal{U}_2^0}{\partial x_3^2} \right\}, \tag{6.19}$$

where $\nu = \dfrac{\lambda}{2(\lambda + \mu)}$ is the Poisson coefficient of the material. Expressions (6.18) and (6.19) permits to derive from (5.36), (5.49)÷(5.51) and (5.54)

$$X_{11} = X_{22} = \nu \left\{ -\frac{\partial \mathcal{U}_3^0}{\partial x_3} + kX_1 \frac{\partial^2 \mathcal{U}_1^0}{\partial x_3^2} + kX_2 \frac{\partial^2 \mathcal{U}_2^0}{\partial x_3^2} \right\}, \tag{6.20}$$

$$X_{12} = 0,$$

$$\Sigma_{11} = \Sigma_{22} = \Sigma_{12} = 0 \text{ a.e. in } \Omega^+ \times D, \tag{6.21}$$

$$\Sigma_{33} = E \left( \frac{\partial \mathcal{U}_3^0}{\partial x_3} - kX_1 \frac{\partial^2 \mathcal{U}_1^0}{\partial x_3^2} - kX_2 \frac{\partial^2 \mathcal{U}_2^0}{\partial x_3^2} \right) \text{ a.e. in } \Omega^+ \times D, \tag{6.22}$$

where $E = \dfrac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$ is the Young modulus of the elastic material.

## 6.2 The rods equations in $\Omega^+$ (case $r_\varepsilon = k\varepsilon$)

Let now $\varphi \in C_0^\infty(\omega)$, $\mathcal{V}_1, \mathcal{V}_2$ be in $C^\infty([0,L])$ such that $\mathcal{V}_1(0) = \mathcal{V}_2(0) = \mathcal{V}_1'(0) = \mathcal{V}_2'(0) = 0$, $\mathcal{A}_3$ be in $C^\infty([0,L])$ such that $\mathcal{A}_3(0) = 0$.

We choose as a test function in (6.1) the field defined in $\Omega_\varepsilon^+$ by

$$
\begin{aligned}
v^\varepsilon(x_1,x_2,x_3) = \varphi(\varepsilon p, \varepsilon q)\Bigg[ & \left(\frac{1}{r_\varepsilon}\mathcal{V}_1(x_3) - \frac{x_2 - \varepsilon q}{r_\varepsilon}\mathcal{A}_3(x_3)\right) e_1 \\
+ \left(\frac{1}{r_\varepsilon}\mathcal{V}_2(x_3) + \frac{x_1 - \varepsilon p}{r_\varepsilon}\mathcal{A}_3(x_3)\right) e_2 + & \left(-\frac{x_1 - \varepsilon p}{r_\varepsilon}\mathcal{V}_1'(x_3) - \frac{x_2 - \varepsilon q}{r_\varepsilon}\mathcal{V}_2'(x_3)\right) e_3\Bigg],
\end{aligned}
\tag{6.23}
$$

if $(x_1,x_2) \in \mathcal{D}_{pq}^\varepsilon$, $x_3 \in ]0,L[$, for $(p,q) \in \mathcal{N}^\varepsilon$, and $v^\varepsilon = 0$ in $\Omega^-$. Remark that the boundary conditions on $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{A}_3$ at $x_3 = 0$ imply that $v^\varepsilon \in H^1(\Omega_\varepsilon)$. Then in $\Omega_\varepsilon^+$ we have

$$
\begin{aligned}
\gamma_{11}(v^\varepsilon) = \gamma_{22}(v^\varepsilon) &= \gamma_{12}(v^\varepsilon) = 0, \\
\gamma_{13}(v^\varepsilon) &= -\varphi(\varepsilon p, \varepsilon q)\frac{x_2 - \varepsilon q}{r_\varepsilon}\frac{1}{2}\mathcal{A}_3'(x_3), \\
\gamma_{23}(v^\varepsilon) &= \varphi(\varepsilon p, \varepsilon q)\frac{x_1 - \varepsilon p}{r_\varepsilon}\frac{1}{2}\mathcal{A}_3'(x_3), \\
\gamma_{33}(v^\varepsilon) &= \varphi(\varepsilon p, \varepsilon q)\left(-\frac{x_1 - \varepsilon p}{r_\varepsilon}\mathcal{V}_1''(x_3) - \frac{x_2 - \varepsilon q}{r_\varepsilon}\mathcal{V}_2''(x_3)\right).
\end{aligned}
$$

With the definition (6.4) of $\widetilde{\varphi}^\varepsilon$ in the previous section, it yields

$$
\begin{aligned}
\mathcal{T}^\varepsilon(\gamma_{\alpha\beta}(v^\varepsilon)) &= 0, \text{ for } \alpha, \beta = 1, 2, \\
\mathcal{T}^\varepsilon(\gamma_{13}(v^\varepsilon)) &= -\widetilde{\varphi}^\varepsilon X_2 \frac{1}{2}\mathcal{A}_3'(x_3), \\
\mathcal{T}^\varepsilon(\gamma_{23}(v^\varepsilon)) &= \widetilde{\varphi}^\varepsilon X_1 \frac{1}{2}\mathcal{A}_3'(x_3), \\
\mathcal{T}^\varepsilon(\gamma_{23}(v^\varepsilon)) &= \widetilde{\varphi}^\varepsilon\left(-X_1\mathcal{V}_1''(x_3) - X_2\mathcal{V}_2''(x_3)\right).
\end{aligned}
$$

Using the convergence (5.21) of $\mathcal{T}^\varepsilon(\sigma_{ij}^\varepsilon)$ allows to pass to the limit in the left hand side of (6.1) to obtain

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{i,j=1}^{3}\int_{\Omega^+\times D}\mathcal{T}^\varepsilon(\sigma_{ij}^\varepsilon)\mathcal{T}^\varepsilon(\gamma_{ij}(v^\varepsilon))\, dx_1dx_2dx_3dX_1dX_2 = \\
&-\int_{\Omega^+\times D}\varphi\Sigma_{13}X_2\mathcal{A}_3'dx_1dx_2dx_3dX_1dX_2 + \int_{\Omega^+\times D}\varphi\Sigma_{23}X_1\mathcal{A}_3'dx_1dx_2dx_3dX_1dX_2 + \\
&\int_{\Omega^+\times D}\varphi\Sigma_{33}\left(-X_1\mathcal{V}_1''(x_3) - X_2\mathcal{V}_2''(x_3)\right)dx_1dx_2dx_3dX_1dX_2.
\end{aligned}
\tag{6.24}
$$

Now, in view of (6.23), we have

$$\mathcal{T}^\varepsilon(v^\varepsilon) = \widetilde{\varphi}^\varepsilon \Bigg[ \left( \frac{1}{r_\varepsilon} \mathcal{V}_1 - X_2 \mathcal{A}_3 \right) e_1 + \left( \frac{1}{r_\varepsilon} \mathcal{V}_2 + X_1 \mathcal{A}_3 \right) e_2 + \left( -X_1 \mathcal{V}_1' - X_2 \mathcal{V}_2' \right) e_3 \Bigg], \tag{6.25}$$

so that with (6.9) and (6.10)

$$\lim_{\varepsilon \to 0} \int_{\Omega^+ \times D} \mathcal{T}^\varepsilon(f^\varepsilon) \mathcal{T}^\varepsilon(v^\varepsilon) dx_1 dx_2 dx_3 dX_1 dX_2 = \int_{\Omega^+ \times D} \varphi \left[ \sum_{\alpha=1}^{2} f_\alpha \mathcal{V}_\alpha + f_3 \left( -X_1 \mathcal{V}_1' - X_2 \mathcal{V}_2' \right) \right] dx_1 dx_2 dx_3 dX_1 dX_2. \tag{6.26}$$

Using (6.15), (6.16) and (6.22), (6.1), (6.24) and (6.26) gives

$$\begin{aligned} &\mu k \int_{\Omega^+ \times D} \varphi (X_1^2 + X_2^2) \frac{\partial \mathcal{R}_3^0}{\partial x_3} \mathcal{A}_3' dx_1 dx_2 dx_3 dX_1 dX_2 + \\ &E \int_{\Omega^+ \times D} \varphi \left[ \frac{\partial \mathcal{U}_3^0}{\partial x_3} - k X_1 \frac{\partial^2 \mathcal{U}_1^0}{\partial x_3^2} - k X_2 \frac{\partial^2 \mathcal{U}_2^0}{\partial x_3^2} \right] \left[ -X_1 \mathcal{V}_1'' - X_2 \mathcal{V}_2'' \right] dx_1 dx_2 dx_3 dX_1 dX_2 = \\ &\int_{\Omega^+ \times D} \varphi \left[ \sum_{\alpha=1}^{2} f_\alpha \mathcal{V}_\alpha + f_3 \left( -X_1 \mathcal{V}_1' - X_2 \mathcal{V}_2' \right) \right] dx_1 dx_2 dx_3 dX_1 dX_2, \end{aligned} \tag{6.27}$$

for any $\varphi \in C_0^\infty(\omega)$, $\mathcal{A}_3 \in C^\infty([0, L])$ such that $\mathcal{A}_3(0) = 0$, for $\mathcal{V}_1, \mathcal{V}_2 \in C^\infty([0, L])$ such that $\mathcal{V}_1(0) = \mathcal{V}_2(0) = \mathcal{V}_1'(0) = \mathcal{V}_2'(0) = 0$.

Taking $\mathcal{V}_1 = \mathcal{V}_2 = 0$ in (6.27) gives together with the boundary condition (5.56)

$$\mathcal{R}_3^0 = 0. \tag{6.28}$$

Once this result is obtained, (6.27) implies that $(\mathcal{U}_1^0, \mathcal{U}_2^0)$ satisfies the equations

$$\begin{cases} kEI_\alpha \dfrac{\partial^4 \mathcal{U}_\alpha^0}{\partial x_3^4} = \pi f_\alpha \text{ a.e. in } \Omega^+, \\ \dfrac{\partial^2 \mathcal{U}_\alpha^0}{\partial x_3^2}(x_1, x_2, L) = \dfrac{\partial^3 \mathcal{U}_\alpha^0}{\partial x_3^3}(x_1, x_2, L) = 0 \text{ a.e. in } \omega, \end{cases} \tag{6.29}$$

for $\alpha = 1, 2$. Recall that in order to obtain (6.29), we have used the fact that

$$\int_D X_1 dX_1 dX_2 = \int_D X_2 dX_1 dX_2 = \int_D X_1 X_2 dX_1 dX_2 = 0.$$

Due to the boundary conditions (5.55) and (5.57), the field $(\mathcal{U}_1^0, \mathcal{U}_2^0)$ is unique in $(L^2(\omega, H^2(]0, L[)))^2$.

## 6.3 The stress transmission condition, the equation for $\mathcal{U}_3^0$ and the equations in $\Omega^-$ (case $r_\varepsilon = k\varepsilon$)

Let us plug an arbitrary test field $v \in \left(C^\infty(\overline{\omega} \times [-l, L])\right)^3$ such that $v = 0$ on $\partial\omega\times]-l, 0[$, in (6.1) (indeed in $\Omega_\varepsilon^+$, $v_{|\Omega_\varepsilon^+} \in (H^1(\Omega_\varepsilon^+))^3$) and we pass to the limit as $\varepsilon$ tends to zero.

To this aim recall first that, by (b) of Lemma 5.1, $\mathcal{T}^\varepsilon(\gamma_{ij}(v)) \to \gamma_{ij}(v)$ strongly in $L^2(\Omega^+ \times D)$ and that $\mathcal{T}^\varepsilon(v) \to v$ strongly in $(L^2(\Omega^+ \times D))^3$. Then using (6.9) and (6.21), it gives

$$
\begin{aligned}
&2k^2 \int_{\Omega^+\times D} \Sigma_{13}\gamma_{13}(v)dx_1dx_2dx_3dX_1dX_2 + 2k^2 \int_{\Omega^+\times D} \Sigma_{23}\gamma_{23}(v)dx_1dx_2dx_3dX_1dX_2+\\
&k^2 \int_{\Omega^+\times D} \Sigma_{33}\gamma_{33}(v)dx_1dx_2dx_3dX_1dX_2 + \int_{\Omega^-} \sum_{i,j=1}^{3} \sigma_{ij}^-\gamma_{ij}(v)dx_1dx_2dx_3 = \\
&k^2 \int_{\Omega^+\times D} f_3v_3dx_1dx_2dx_3dX_1dX_2 + \int_{\Omega^-} \sum_{i=1}^{3} f_iv_idx_1dx_2dx_3,
\end{aligned}
\tag{6.30}
$$

where

$$
\sigma^- = \lambda\left(\operatorname{Tr}\gamma(u^-)\right)I + 2\mu\gamma(u^-) \in \left(L^2(\Omega^-)\right)^{3\times 3}. \tag{6.31}
$$

Now, because of (6.15), (6.16) and (6.28) the two first terms of (6.30) are equal to zero. Moreover, the expression (6.22) of $\Sigma_{33}$ permits to obtain from (6.30) (note that $\gamma_{13}$ does not depend on $(X_1, X_2)$)

$$
\begin{aligned}
&Ek^2\pi \int_{\Omega^+} \frac{\partial\mathcal{U}_3^0}{\partial x_3}\frac{\partial v_3}{\partial x_3}dx_1dx_2dx_3 + \int_{\Omega^-} \sum_{i,j=1}^{3} \sigma_{ij}^-\gamma_{ij}(v)dx_1dx_2dx_3 = \\
&k^2\pi \int_{\Omega^+} f_3v_3dx_1dx_2dx_3 + \int_{\Omega^-} \sum_{i=1}^{3} f_iv_idx_1dx_2dx_3,
\end{aligned}
\tag{6.32}
$$

for any $v \in \left(C^\infty(\overline{\omega} \times [-l, L])\right)^3$ such that $v = 0$ on $\partial\omega\times]-l, 0[$.

If $W$ is the Hilbert space defined by

$$
\begin{aligned}
&W = \{(\mathcal{V}, v) \in L^2(\omega, H^1(]0, L[)) \times (H^1(\Omega^-))^3;\\
&\mathcal{V}(x_1, x_2, 0) = v_3(x_1, x_2, 0) \text{ on } \omega \text{ and } v = 0 \text{ on } \partial\omega\times]-l, 0[\},
\end{aligned}
\tag{6.33}
$$

the continuity condition (5.59) shows that $(\mathcal{U}_3^0, u^-) \in W$. Then Korn's inequality in $\Omega^-$ (together with the expression (6.31) of $\sigma^-$) implies that (6.32) (which indeed holds true for any $v \in W$ by density) admits a unique solution $(\mathcal{U}_3^0, u^-) \in W$. In terms of equations on $\Omega^+$ and $\Omega^-$ and of a transmission condition and boundary conditions, it gives

$$
-E\frac{\partial^2\mathcal{U}_3^0}{\partial x_3^2} = f_3 \text{ in } \Omega^+ \tag{6.34}
$$

$$-\sum_{j=1}^{3}\partial_j\sigma^-_{ij} = f_i \text{ in } \Omega^-, \text{ for } i=1,2,3, \tag{6.35}$$

$$\sigma^-_{33} = Ek^2\pi\frac{\partial\mathcal{U}^0_3}{\partial x_3} \text{ on } \omega, \tag{6.36}$$

$$\frac{\partial\mathcal{U}^0_3}{\partial x_3} = 0 \text{ on } \omega\times\{L\}, \tag{6.37}$$

$$\sigma^-_{\alpha 3} = 0 \text{ on } \omega \text{ and on } \omega\times\{-l\}, \tag{6.38}$$

$$\sigma^-_{33} = 0 \text{ on } \omega \text{ and on } \omega\times\{-l\}. \tag{6.39}$$

Equation (6.34) is the standard compression-traction equation for rods and here $(x_1, x_2)$ appears as a parameter (as this was the case for (6.29)). In some sense, the rods equations (6.29) describe a continuum of rods indexed by $(x_1, x_2) \in \omega$.

Equations (6.35) together the constitutive law (6.31) are the standard equations of elasticity in $\Omega^-$. The equation (6.36) reflects the continuity of the normal stress between the rods and $\Omega^-$ since it can be written as

$$\sigma^-_{33} = k^2\int_D \Sigma_{33}dX_1dX_2, \text{ on } \omega.$$

# 7 The case $\dfrac{r_\varepsilon}{\varepsilon} \to 0$

We start with the estimates of Lemma 4.2 and Lemma 5.3 which are still valid in the case $\dfrac{r_\varepsilon}{\varepsilon} \to 0$. By comparison with the analysis performed in the preceding sections for the case $r_\varepsilon = k\varepsilon$, those estimates show that a few fields must be re-scaled (e.g. by multiplication by $\dfrac{r_\varepsilon}{\varepsilon}$) to exhibit weak limits. Once these re-scalings are adopted, many points of the analysis are identical in both cases. As a consequence, we will only detail the points where the arguments must be modified.

## 7.1 Weak limits of the fields (case $\dfrac{r_\varepsilon}{\varepsilon} \to 0$)

Lemma 4.2 and Lemma 5.3 give the following weak convergence results:

**Lemma 7.1.** *Assume (2.19)÷(2.21), and that* $\dfrac{r_\varepsilon}{\varepsilon} \to 0$.

*For a subsequence, still denoted by* $\{\varepsilon\}$,

• *there exist* $u^0_i \in L^2(\omega, H^1(D\times]0,L[))$ *and* $\overline{u}^0_i \in L^2(\Omega^+, H^1(D))$, *for* $i = 1,2,3$, *such that, as* $\varepsilon$ *tends to zero,*

$$\frac{r^2_\varepsilon}{\varepsilon}\mathcal{T}^\varepsilon(u^\varepsilon_\alpha) \rightharpoonup u^0_\alpha \text{ weakly in } L^2(\omega, H^1(D\times]0,L[)), \text{ for } \alpha = 1,2, \tag{7.1}$$

$$\frac{r_\varepsilon}{\varepsilon}\mathcal{T}^\varepsilon(u^\varepsilon_3) \rightharpoonup u^0_3 \text{ weakly in } L^2(\omega, H^1(D\times]0,L[)), \tag{7.2}$$

$$\frac{1}{\varepsilon}\mathcal{T}^{\varepsilon}(\overline{u}_i^{\varepsilon}) \rightharpoonup \overline{u}_i^0 \text{ weakly in } L^2(\Omega^+, H^1(D)), \text{ for } i = 1, 2, 3; \tag{7.3}$$

• *there exist* $\mathcal{U}_i^0 \in L^2(\omega, H^1(]0, L[))$, $\mathcal{R}_i^0 \in L^2(\omega, H^1(]0, L[))$, *for* $i = 1, 2, 3$, *and* $Z \in (L^2(\Omega^+))^3$ *such that, as* $\varepsilon$ *tends to zero,*

$$\frac{r_\varepsilon^2}{\varepsilon}\mathcal{U}_\alpha^{\varepsilon} \rightharpoonup \mathcal{U}_\alpha^0 \text{ weakly in } L^2\left(\omega, H^1(]0, L[)\right), \text{ for } \alpha = 1, 2, \tag{7.4}$$

$$\frac{r_\varepsilon}{\varepsilon}\mathcal{U}_3^{\varepsilon} \rightharpoonup \mathcal{U}_3^0 \text{ weakly in } L^2\left(\omega, H^1(]0, L[)\right), \tag{7.5}$$

$$\frac{r_\varepsilon^2}{\varepsilon}\mathcal{R}_i^{\varepsilon} \rightharpoonup \mathcal{R}_i^0 \text{ weakly in } L^2\left(\omega, H^1(]0, L[)\right), \text{ for } i = 1, 2, 3, \tag{7.6}$$

$$\frac{r_\varepsilon}{\varepsilon}\left(\frac{\partial \mathcal{U}^{\varepsilon}}{\partial x_3} - (\mathcal{R}^{\varepsilon} \wedge e_3)\right) \rightharpoonup Z \text{ weakly in } \left(L^2(\Omega^+)\right)^3; \tag{7.7}$$

• *there exist* $X_{ij} \in L^2(\Omega^+ \times D)$ *and* $\Sigma_{ij} \in L^2(\Omega^+ \times D)$, *for* $i, j = 1, 2, 3$, *such that, as* $\varepsilon$ *tends to zero,*

$$\frac{r_\varepsilon}{\varepsilon}\mathcal{T}^{\varepsilon}(\gamma_{ij}(u^{\varepsilon})) \rightharpoonup X_{ij} \text{ weakly in } L^2(\Omega^+ \times D), \text{ for } i, j = 1, 2, 3, \tag{7.8}$$

$$\frac{r_\varepsilon}{\varepsilon}\mathcal{T}^{\varepsilon}(\sigma_{ij}^{\varepsilon}) \rightharpoonup \Sigma_{ij} \text{ weakly in } L^2(\Omega^+ \times D), \text{ for } i, j = 1, 2, 3; \tag{7.9}$$

• *there exist* $u_i^- \in L^2(\Omega^-)$, *with* $u_i^- = 0$ *on* $\partial\omega \times ] - l, 0[$, *for* $i = 1, 2, 3$, *such that, as* $\varepsilon$ *tends to zero,*

$$u_i^{\varepsilon} \rightharpoonup u_i^- \text{ weakly in } H^1(\Omega^-) \text{ strongly in } L^2(\Omega^-). \tag{7.10}$$

With the limit introduced in Lemma 7.1, the analysis developed in Section 5.3 remains identical so that $\mathcal{U}_i^0$, $u_i^0$ and $\Sigma_{ij}$ verify (5.23) (5.24), (5.32) and (5.49)÷(5.54) in $\Omega^+ \times D$ with $k = 1$, and the boundary conditions (5.55)÷(5.57). Let us just explain why $k$ becomes 1 (and not 0) in those expressions. Loosely speaking, when the unfolding operator $\mathcal{T}^{\varepsilon}$ is applied to a field it results that some terms are multiplied by $r$ (see e.g. (5.27)). Then, in the case where $r_\varepsilon = k\varepsilon$, the corresponding weak limits are multiplied by $k$. In the case where $\frac{r_\varepsilon}{\varepsilon} \to 0$, the rescaling of the fields (as shown in Lemma 7.1) leads to the constant 1 when $\mathcal{T}^{\varepsilon}$ is applied. Let us now turn to the analog of the kinematic conditions obtained in Section 5.5. Conditions (5.55) and (5.56) remain true and are derived identically. The main difference here is the continuity condition (5.59) which can not be established here, because the measure of the set $\Omega_\varepsilon^+$ goes to zero too rapidly. Defining $\mathcal{T}^{\varepsilon}(u_3^{\varepsilon})$ as in Section 5.5 also for $x_3 \in ]-l, 0[$, we have here $\frac{r_\varepsilon}{\varepsilon}\mathcal{T}^{\varepsilon}(u_3^{\varepsilon})$ bounded in $L^2(\omega \times D, H^1(]-l, L[))$ (because of estimates of Lemma 5.3). Then $\frac{r_\varepsilon}{\varepsilon}\mathcal{T}^{\varepsilon}(u_3^{\varepsilon}) \rightharpoonup u_3^*$ weakly in $L^2(\omega \times D, H^1(]-l, L[))$ (for a subsequence) as $\varepsilon$ tends to zero. Because of the weak convergence of $\frac{r_\varepsilon}{\varepsilon}\mathcal{T}^{\varepsilon}(u_3^{\varepsilon})$ in Lemma 7.1 and using (5.31) which holds true with $k = 1$ in the present case, we obtain

$$u_3^* = \mathcal{U}_3^0 + X_2\mathcal{R}_1^0 - X_1\mathcal{R}_2^0 \text{ in } \Omega^+ \times D. \tag{7.11}$$

Now, from (7.10) we know that $u_3^{\varepsilon} \to u_3^-$ strongly in $L^2(\Omega^-)$. As we have

$\dfrac{r_\varepsilon}{\varepsilon}\|\mathcal{T}^\varepsilon(u_3^\varepsilon)\|_{L^2(\Omega^-\times D)} = \left(\displaystyle\int_{\omega_{\varepsilon,r_\varepsilon}\times]-l,0[}|u_3^\varepsilon|^2\right)^{\frac{1}{2}}$, we deduce that $\dfrac{r_\varepsilon}{\varepsilon}\mathcal{T}^\varepsilon(u_3^\varepsilon) \to 0$ strongly in $L^2(\Omega^-\times D)$. Then $u_3^*(x_1,x_2,0,X_1,X_2)=0$ in $L^2(\omega\times D)$ which implies with (7.11) that

$$\mathcal{U}_3^0(x_1,x_2,0)=0 \text{ on } \omega. \tag{7.12}$$

Next deriving the kinematic conditions (5.67), (5.68) and (5.69) on $\overline{u}^0$ is identical to the case $r_\varepsilon = k\varepsilon$. We now turn to obtaining the limit problem. Writing (2.18) in terms of the operator $\mathcal{T}^\varepsilon$ gives here (see (a) of Lemma 5.1)

$$\begin{aligned}
&\frac{r_\varepsilon^2}{\varepsilon^2}\sum_{i,j=1}^3\int_{\Omega^+\times D}\mathcal{T}^\varepsilon\left(\sigma_{ij}^\varepsilon\right)\mathcal{T}^\varepsilon\left(\gamma_{ij}(v)\right)dx_1dx_2dx_3dX_1dX_2+\\
&\sum_{i,j=1}^3\int_{\Omega^-}\sigma_{ij}^\varepsilon\gamma_{ij}(v)dx_1dx_2dx_3=\\
&\frac{r_\varepsilon^2}{\varepsilon^2}\sum_{i=1}^3\int_{\Omega^+\times D}\mathcal{T}^\varepsilon\left(f_i^\varepsilon\right)\mathcal{T}^\varepsilon\left(v_i\right)dx_1dx_2dx_3dX_1dX_2\\
&+\sum_{i=1}^3\int_{\Omega^-}f_i^\varepsilon v_i dx_1dx_2dx_3, \quad \forall v\in V_{\varepsilon,r}.
\end{aligned} \tag{7.13}$$

Recall that we have by assumptions (2.19) and (2.20)

$$\mathcal{T}^\varepsilon(f_\alpha^\varepsilon) = r_\varepsilon\mathcal{T}^\varepsilon(f_\alpha), \text{ for } \alpha=1,2,$$

and

$$\mathcal{T}^\varepsilon(f_3^\varepsilon)=\mathcal{T}^\varepsilon(f_3),$$

and by (a) of Lemma 5.1

$$\frac{r_\varepsilon}{\varepsilon}\|\mathcal{T}^\varepsilon(f_i^\varepsilon)\|_{L^2(\Omega^+\times D)} = \|f_i\|_{L^2(\Omega_\varepsilon^+)} \text{ for } i=1,2,3.$$

Then

$$\frac{1}{\varepsilon}\|\mathcal{T}^\varepsilon(f_\alpha^\varepsilon)\|_{L^2(\Omega^+\times D)} = \|f_\alpha\|_{L^2(\Omega_\varepsilon^+)} \to 0 \text{ for } \alpha=1,2, \tag{7.14}$$

and

$$\frac{r_\varepsilon}{\varepsilon}\|\mathcal{T}^\varepsilon(f_3^\varepsilon)\|_{L^2(\Omega^+\times D)} = \|f_3\|_{L^2(\Omega_\varepsilon^+)} \to 0, \tag{7.15}$$

because $f_i\in L^2(\Omega^+)$ for i=1,2,3 and meas$(\Omega_\varepsilon^+)\to 0$.

As far as the determination of $\overline{u}^0$ is concerned, we choose the same test functions $v^\varepsilon$ given by (6.2) and (6.3) in (7.13). With the help of the convergence on $\dfrac{r_\varepsilon}{\varepsilon}\mathcal{T}^\varepsilon\left(\sigma^\varepsilon\right)$ given by Lemma 7.1 and of (7.14), (7.15) we obtain the same problem (6.13) and (6.17) with $k=1$. It turns out that $\overline{u}_3^0 = 0$, and that $\overline{u}_1^0$ and $\overline{u}_2^0$ are given by (6.18), (6.19), and that $\Sigma$ is given by (6.15), (6.16), (6.21) and (6.22).

To obtain the rods equations in $\Omega^+$, we first use the function $v^\varepsilon$ defined in (6.23) as a test function in (7.13). Taking into account (7.14), (7.15), we deduce that (6.27) holds true with a right hand side equal to zero. It follows that (6.28) and (6.29) are satisfied with a right hand side equal to zero and with $k = 1$. In view of the boundary condition satisfied by $\mathcal{U}_\alpha^0$ on $\partial\omega$, we obtain $\mathcal{U}_\alpha^0 = 0$ in $\Omega^+ \times D$.

In order to obtain the equation for $\mathcal{U}_3^0$ in $\Omega^+$, we choose in (7.13) the test function $v^\varepsilon$ defined by

$$v^\varepsilon(x_1, x_2, x_3) = \varphi(\varepsilon p, \varepsilon q)\mathcal{V}_3\,(x_3)\, e_3, \text{ if } (x_1, x_2) \in \mathcal{D}^\varepsilon_{pq},\ x_3 \in ]0, L[, \text{ for } (p, q) \in \mathcal{N}^\varepsilon,$$

and

$$v^\varepsilon(x_1, x_2, x_3) = 0, \text{ if } x_3 \in ] - l, 0[.$$

where $\varphi$ is in $C_0^\infty(\omega)$ and $\mathcal{V}_3 \in C^\infty([0, L])$ with $\mathcal{V}_3(0) = 0$. Then we have in $\Omega_\varepsilon^+$:

$$\gamma_{ij}(v^\varepsilon) = 0, \text{ for } (i, j) \neq (3, 3),$$

$$\gamma_{33}(v^\varepsilon) = \varphi(\varepsilon p, \varepsilon q)\mathcal{V}_3'\,(x_3)\,, \text{ if } (x_1, x_2) \in \mathcal{D}^\varepsilon_{pq},\ x_3 \in ]0, L[, \text{ for } (p, q) \in \mathcal{N}^\varepsilon.$$

Using the same type of arguments than in Section 6.2, we obtain

$$\mathcal{T}^\varepsilon(v^\varepsilon) \to \varphi\mathcal{V}_3 \text{ in } L^2(\Omega^+ \times D),$$

$$\mathcal{T}^\varepsilon(\gamma_{ij}(v^\varepsilon)) = 0, \quad (i, j) \neq (3, 3),$$
$$\mathcal{T}^\varepsilon(\gamma_{33}(v^\varepsilon)) \to \varphi\mathcal{V}_3',\ \text{in } L^2(\Omega^+ \times D),$$

as $\varepsilon$ tends to zero.

With the help of the weak convergence of $\dfrac{r_\varepsilon}{\varepsilon}\mathcal{T}^\varepsilon(\sigma_{ij})$, (7.14), (7.15), (6.22) and the fact that $\mathcal{U}_1^0 = \mathcal{U}_2^0 = 0$, we pass to the limit in (7.13) and it yields

$$\int_{\Omega^+} \frac{\partial \mathcal{U}_3^0}{\partial x_3}\mathcal{V}_3' dx_1 dx_2 dx_3 = 0, \tag{7.16}$$

for any $\varphi \in C_0^\infty(\omega)$ and $\mathcal{V}_3 \in C^\infty([0, L])$ with $\mathcal{V}_3(0) = 0$. Indeed (7.16) gives equation and

$$\frac{\partial^2 \mathcal{U}_3^0}{\partial x_3^2} = 0, \text{ a.e. in } L^2(\Omega^+),$$

and

$$\frac{\partial \mathcal{U}_3^0}{\partial x_3} = 0, \text{ in } \omega \times \{L\},$$

and because of the boundary condition (7.12), it follows that $\mathcal{U}_3^0 = 0$ in $\Omega^+$. In conclusion, in the present case where $\dfrac{r_\varepsilon}{\varepsilon} \to 0$, we find that $\mathcal{U}_i^0 = 0$ in $\Omega^+$, for $i = 1, 2, 3$. To conclude this section it remains to obtain the equations and boundary conditions in $\Omega^-$. To this end, let us take $v \in C^\infty(\overline{\omega} \times [-l, L])$ such that $v = 0$ on $\partial\omega\times] - l, 0[$ as a test function in (7.13). With the help of (c) of Lemma 5.1, we have $\mathcal{T}^\varepsilon(\gamma_{ij}(v)) \to \gamma_{ij}(v)$ strongly in $L^2(\Omega^+ \times D)$, for

$i = 1, 2, 3$, and $\mathcal{T}^\varepsilon(v) \to v$ strongly in $L^2(\Omega^+ \times D)$, as $\varepsilon$ tends to zero. In view of the weak convergence of $\frac{r_\varepsilon}{\varepsilon}\mathcal{T}^\varepsilon(\sigma^\varepsilon)$ given by Lemma 7.1 and of (7.14), (7.15), passing to the limit in (7.13) leads to

$$\sum_{i,j=1}^{3} \int_{\Omega^-} \sigma_{ij}^- \gamma_{ij} dx_1 dx_2 dx_3 = \sum_{i=1}^{3} \int_{\Omega^-} f_i v_i dx_1 dx_2 dx_3,$$

for any $v$ as above. Then, we obtain

$$-\sum_{j=1}^{3} \frac{\partial \sigma_{ij}^-}{\partial x_j} = f_i \text{ in } \Omega^-,$$

$$\sigma_{\alpha 3}^- = \sigma_{33}^- = 0, \text{ on } \omega \times \{0\} \text{ and } \omega \times \{-l\}.$$

Since $\sigma_{ij}^-$ is still given by (6.31), it gives a standard elastic problem in $\Omega^-$ which indeed admits a unique solution.

# 8 Convergence of the energies

We only investigate the case $r = k\varepsilon$, the case $\frac{r_\varepsilon}{\varepsilon} \to 0$ being very similar. We take $v = u^\varepsilon$ in (6.1) to obtain the energy identity:

$$\mathcal{E}_{\Omega_\varepsilon}(u^\varepsilon) = k^2 \sum_{i,j=1}^{3} \int_{\Omega^+ \times D} \mathcal{T}^\varepsilon\left(\sigma_{ij}^\varepsilon\right) \mathcal{T}^\varepsilon\left(\gamma_{ij}(u^\varepsilon)\right) dx_1 dx_2 dx_3 dX_1 dX_2 +$$

$$\sum_{i,j=1}^{3} \int_{\Omega^-} \sigma_{ij}^\varepsilon \gamma_{ij}(u^\varepsilon) dx_1 dx_2 dx_3 = k^2 \sum_{i=1}^{3} \int_{\Omega^+ \times D} \mathcal{T}^\varepsilon\left(f_i^\varepsilon\right) \mathcal{T}^\varepsilon\left(u_i^\varepsilon\right) dx_1 dx_2 dx_3 dX_1 dX_2$$

$$+ \sum_{i=1}^{3} \int_{\Omega^-} f_i^\varepsilon u_i^\varepsilon dx_1 dx_2 dx_3.$$

Since $r_\varepsilon = k\varepsilon$, from (2.19) and (2.20) we have

$$k^2 \sum_{i=1}^{3} \int_{\Omega^+ \times D} \mathcal{T}^\varepsilon\left(f_i^\varepsilon\right) \mathcal{T}^\varepsilon\left(u_i^\varepsilon\right) dx_1 dx_2 dx_3 dX_1 dX_2 =$$

$$k^3 \sum_{\alpha=1}^{2} \int_{\Omega^+ \times D} \varepsilon \mathcal{T}^\varepsilon\left(f_\alpha\right) \mathcal{T}^\varepsilon\left(u_\alpha^\varepsilon\right) dx_1 dx_2 dx_3 dX_1 dX_2 +$$

$$k^2 \int_{\Omega^+ \times D} \mathcal{T}^\varepsilon\left(f_3\right) \mathcal{T}^\varepsilon\left(u_3^\varepsilon\right) dx_1 dx_2 dx_3 dX_1 dX_2,$$

and (5.13), (5.14) and the strong convergence of $\mathcal{T}^\varepsilon(f_i)$ to $f_i$ permits us to obtain

$$\lim_{\varepsilon\to 0}\mathcal{E}_{\Omega_\varepsilon}(u^\varepsilon) = k^3\sum_{\alpha=1}^{2}\int_{\Omega^+\times D} f_\alpha u_\alpha^0 dx_1dx_2dx_3dX_1dX_2 + \\ k^2\int_{\Omega^+\times D} f_3u_3^0 dx_1dx_2dx_3dX_1dX_2 + \sum_{i=1}^{3}\int_{\Omega^-} f_iu_i^- dx_1dx_2dx_3. \tag{8.1}$$

Now remark that the expressions derived in the preceding section lead to

$$X_{11}+X_{22}+2\nu X_{33}=0, \qquad X_{11}=X_{22}, \qquad X_{12}=X_{13}=X_{23}=0, \tag{8.2}$$

$$X_{33}=\frac{\partial\mathcal{U}_3^0}{\partial x_3} - kX_1\frac{\partial^2\mathcal{U}_1^0}{\partial x_3^2} - kX_2\frac{\partial^2\mathcal{U}_2^0}{\partial x_3^2}. \tag{8.3}$$

We will now use the easy to verify algebraic identity which is valid for any symmetric matrix $d=(d_{ij})$

$$\lambda(\mathrm{Tr}\ d)(\mathrm{Tr}\ d)+2\mu\sum_{i,j=1}^{3} d_{ij}d_{ij} = Ed_{33}d_{33} + \frac{E}{(1+\nu)(1-2\nu)}(d_{11}+d_{22}+2\nu d_{33})^2 + \\ \frac{E}{2(1+\nu)}\left[(d_{11}-d_{22})^2+4(d_{12}^2+d_{13}^2+d_{23}^2)\right]. \tag{8.4}$$

Then we have, in view of (8.2) and (8.3),

$$\int_{\Omega^+\times D}\left\{\lambda\mathrm{Tr}(X)\mathrm{Tr}(X)+\sum_{i,j=1}^{3}2\mu X_{ij}X_{ij}\right\}dx_1dx_2dx_3dX_1dX_2 = \\ E\int_{\Omega^+\times D}\left(\frac{\partial\mathcal{U}_3^0}{\partial x_3} - kX_1\frac{\partial^2\mathcal{U}_1^0}{\partial x_3^2} - kX_2\frac{\partial^2\mathcal{U}_2^0}{\partial x_3^2}\right)^2 dx_1dx_2dx_3dX_1dX_2 = \\ E\pi\int_{\Omega^+}\left(\frac{\partial\mathcal{U}_3^0}{\partial x_3}\right)^2 dx_1dx_2dx_3 + \\ Ek^2\int_{\Omega^+}\left(I_1\left(\frac{\partial^2\mathcal{U}_1^0}{\partial x_3^2}\right)^2 + I_2\left(\frac{\partial^2\mathcal{U}_2^0}{\partial x_3^2}\right)^2\right)dx_1dx_2dx_3, \tag{8.5}$$

(using again $\int_D X_1dX_1dX_2=\int_D X_2dX_1dX_2=\int_D X_1X_2dX_1dX_2=0$).

Using $\mathcal{U}_\alpha^0=u_\alpha^0$ (recall (5.28)) as a test function in (6.29) gives also taking into account the boundary conditions on $\mathcal{U}_\alpha^0$:

$$
\begin{aligned}
&Ek^4 \int_{\Omega^+} \left( I_1 \left( \frac{\partial^2 \mathcal{U}_1^0}{\partial x_3^2} \right)^2 + I_2 \left( \frac{\partial^2 \mathcal{U}_2^0}{\partial x_3^2} \right)^2 \right) dx_1 dx_2 dx_3 = \\
&k^3 \sum_{\alpha=1}^{2} \int_{\Omega^+} f_\alpha u_\alpha^0 dx_1 dx_2 dx_3
\end{aligned}
\tag{8.6}
$$

Secondly, plugging the test function defined by $\mathcal{U}_3^0$ in $\Omega^+ \times D$ and $u^-$ in $\Omega^-$ in (6.32), and recalling (5.32), leads to

$$
\begin{aligned}
&Ek^2\pi \int_{\Omega^+} \left( \frac{\partial \mathcal{U}_3^0}{\partial x_3} \right)^2 dx_1 dx_2 dx_3 + \int_{\Omega^-} \sum_{ij=1}^{3} \sigma_{ij}^- \gamma_{ij}(u^-) dx_1 dx_2 dx_3 = \\
&k^2 \int_{\Omega^+ \times D} f_3 u_3^0 dx_1 dx_2 dx_3 dX_1 dX_2 + \int_{\Omega^-} \sum_{i=1}^{3} f_i u_i^- dx_1 dx_2 dx_3.
\end{aligned}
\tag{8.7}
$$

Adding (8.6) and (8.7) and using (8.1) and (8.5) give

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \mathcal{E}_{\Omega_\varepsilon}(u^\varepsilon) = k^2 \int_{\Omega^+ \times D} \left\{ \lambda \mathrm{Tr}(X) \mathrm{Tr}(X) + \sum_{i,j=1}^{3} 2\mu X_{ij} X_{ij} \right\} dx_1 dx_2 dx_3 dX_1 dX_2 \\
&+ \int_{\Omega^-} \sum_{ij=1}^{3} \sigma_{ij}^- \gamma_{ij}(u^-) dx_1 dx_2 dx_3,
\end{aligned}
\tag{8.8}
$$

which yields the convergence of the energy $\mathcal{E}(u^\varepsilon)$ to the elastic limit energy. A standard argument based on the strict convexity of the elastic energy shows that the convergences (5.20) and (5.21) are strong in $L^2(\Omega^+ \times D)$ and that $\gamma_{ij}(u^\varepsilon) \to \gamma_{ij}(u^-)$ strongly in $L^2(\Omega^-)$ as $\varepsilon$ tends to zero. This last fact implies directly that $u^\varepsilon \to u^-$ strongly in $H^1(\Omega^-)$.

The strong convergence in (5.20), for $i = j = 3$, together with (5.15) and the expression (5.47) of $X_{33}$ gives

$$
\begin{aligned}
&\frac{\partial \mathcal{U}_3^\varepsilon}{\partial x_3} + k\varepsilon X_2 \frac{\partial \mathcal{R}_1^\varepsilon}{\partial x_3} - k\varepsilon X_1 \frac{\partial \mathcal{R}_2^\varepsilon}{\partial x_3} + \frac{\partial \mathcal{T}^\varepsilon(\overline{u}_3^\varepsilon)}{\partial x_3} \to \frac{\partial \mathcal{U}_3^0}{\partial x_3} + kX_2 \frac{\partial \mathcal{R}_1^0}{\partial x_3} - kX_1 \frac{\partial \mathcal{R}_2^0}{\partial x_3} \\
&\qquad\qquad \text{strongly in } L^2(\Omega^+ \times D),
\end{aligned}
\tag{8.9}
$$

as $\varepsilon$ tends to zero. Using $\displaystyle\int_D \mathcal{T}^\varepsilon(\overline{u}_3^\varepsilon) dX_1 dX_2 = \int_D X_\alpha \mathcal{T}^\varepsilon(\overline{u}_3^\varepsilon) dX_1 dX_2 = 0$ a.e. in $\Omega^+$, for $\alpha = 1, 2$, we easily deduce from (8.9) that

$$
\frac{\partial \mathcal{U}_3^\varepsilon}{\partial x_3} \to \frac{\partial \mathcal{U}_3^0}{\partial x_3}, \quad \varepsilon \frac{\partial \mathcal{R}_\alpha^\varepsilon}{\partial x_3} \to \frac{\partial \mathcal{R}_\alpha^0}{\partial x_3} \text{ strongly in } L^2(\Omega^+ \times D), \quad \text{for } \alpha = 1, 2,
\tag{8.10}
$$

as $\varepsilon$ tends to zero.

Now remark that, in view of (4.25) we know that $\varepsilon\mathcal{R}^\varepsilon_\alpha(\cdot,\cdot,0)$ strongly converges to 0 in $L^2(\omega)$, as $\varepsilon$ tends to zero. Then (8.10) implies that

$$\varepsilon\mathcal{R}^\varepsilon_\alpha \to \mathcal{R}^0_\alpha \text{ strongly in } L^2(\omega; H^1(]0,L[)), \quad \text{for } \alpha=1,2, \tag{8.11}$$

as $\varepsilon$ tends to zero. From (5.19) and (8.11) it follows that

$$\varepsilon\mathcal{U}^\varepsilon_\alpha \to \mathcal{U}^0_\alpha \text{ strongly in } L^2(\omega; H^1(]0,L[)), \quad \text{for } \alpha=1,2, \tag{8.12}$$

as $\varepsilon$ tends to zero. As a consequence of the decomposition 4.38, 4.39 of the $u^\varepsilon_\alpha$'s, we deduce from the previous convergences that

$$\varepsilon\mathcal{T}^\varepsilon(u^\varepsilon_\alpha) \to \mathcal{U}^0_\alpha \text{ strongly in } L^2(\omega; H^1(D\times]0,L[)), \quad \text{for } \alpha=1,2. \tag{8.13}$$

As far as $\mathcal{U}^\varepsilon_3$ is concerned, remark that $u^\varepsilon_3(x_1,x_2,0)\to u^0_3(x_1,x_2,0)$ strongly in $L^2(\omega)$ and then the estimates on $\mathcal{U}^\varepsilon_3(x_1,x_2,0)$ in Step 1 of Section 4.1 shows that $\mathcal{U}^\varepsilon_3(x_1,x_2,0)\to\mathcal{U}^0_3(x_1,x_2,0)$ strongly in $L^2(\omega)$. With (8.10) it gives

$$\mathcal{U}^\varepsilon_3 \to \mathcal{U}^0_3 \text{ strongly in } L^2(\omega; H^1(]0,L[)), \tag{8.14}$$

as $\varepsilon$ tends to zero. At least, proceeding as above leads to

$$\mathcal{T}^\varepsilon(u^\varepsilon_3) \to \mathcal{U}^0_3 - kX_1\frac{\partial\mathcal{U}^0_1}{\partial x_3} - kX_2\frac{\partial\mathcal{U}^0_2}{\partial x_3} \text{ strongly in } L^2(\omega; H^1(D\times]0,L[)). \tag{8.15}$$

**Remark 8.1.** *As far as the strong convergences of the sequences* $\frac{1}{\varepsilon}\mathcal{T}^\varepsilon(\overline{u}^\varepsilon)$ *and* $\varepsilon\mathcal{R}^\varepsilon_3$ *in (5.15) and (5.18) are concerned, the analysis is more intricate (even for a single rod, see [16]). What is easy to prove is that* $\frac{1}{\varepsilon}\mathcal{T}^\varepsilon(\overline{u}^\varepsilon_\alpha)\to\overline{u}^0_\alpha$ *strongly in* $L^2(\Omega^+; H^1(D))$ *for* $\alpha=1,2$. *This is a consequence of the strong convergence of* $\mathcal{T}^\varepsilon(\gamma_{\alpha\beta}(u^\varepsilon))$ *in* $L^2(\Omega^+\times D)$, *of (5.34) and of the Korn's inequality in* $D$ *for a displacement field satisfying (3.8) and (3.10).*

# 9 Summarize (case $r = k\varepsilon$)

Let $\varepsilon$ be a sequence of positive real numbers which tends to 0. Let $(u^\varepsilon,\sigma^\varepsilon)$ be the solution of (2.13)÷(2.18) and $\mathcal{U}^\varepsilon$ and $\mathcal{R}^\varepsilon$ be the two first terms of the decomposition of $u^\varepsilon$ in $\Omega^+_\varepsilon$ given in Section 3. The unfolding operator $\mathcal{T}^\varepsilon$ in $\Omega^+_\varepsilon$ is defined in Section 5.1.

In order to state the convergence theorem below, we first recall the limit problems obtained in Section 6.2 and 6.3.

Limit problem: let $(f_1,f_2,f_3)$ be in $(L^2(\Omega))^3$.

• Bending problem in the rods (indexed $(x_1,x_2)\in\omega$):

Let us denote by $(\mathcal{U}_1^0, \mathcal{U}_1^0) \in \left(L^2(\omega; H^1(]0,L[))\right)^2$ be the unique weak solution of the problem:

$$\begin{cases} kEI_\alpha \dfrac{\partial^4 \mathcal{U}_\alpha^0}{\partial x_3^4} = \pi f_\alpha, \text{ in } \Omega^+, \\ \\ \mathcal{U}_\alpha^0 = \dfrac{\partial \mathcal{U}_\alpha^0}{\partial x_3} \text{ in } \omega \times \{0\}, \\ \\ \dfrac{\partial^2 \mathcal{U}_\alpha^0}{\partial x_3^2} = \dfrac{\partial^3 \mathcal{U}_\alpha^0}{\partial x_3^3} = 0 \text{ in } \omega \times \{L\}. \end{cases} \tag{9.1}$$

• Coupled problem for the stretching in the rods and 3d elasticity in $\Omega^-$: let us denote by $\mathcal{U}_3^0 \in (L^2(\omega; H^1(]0,L[)))$ and $(u^-, \sigma^-) \in \left(H^1(\Omega^-)\right)^3 \times \left(L^2(\Omega)\right)_s^{3\times 3}$ the unique weak solution of the problem:

$$\begin{cases} -E \dfrac{\partial^2 \mathcal{U}_3^0}{\partial x_3^2} = f_3, \text{ in } \Omega^+, \\ \\ \sigma_{ij}^- = \lambda \left( \displaystyle\sum_{k=1}^{3} \gamma_{kk}(u^-) \right) \delta_{ij} + 2\mu \gamma_{ij}(u^-) \text{ in } \Omega^-, \\ \\ -\displaystyle\sum_{j=1}^{3} \dfrac{\partial \sigma_{ij}^-}{\partial x_j} = f_i \text{ in } \Omega^- \end{cases} \tag{9.2}$$

with

• the transmission condition on $\omega \times \{0\}$:

$$\begin{cases} \mathcal{U}_3 = u_3^- \text{ on } \omega \times \{0\}, \\ \\ \sigma_{\alpha 3}^- = 0, \quad \sigma_{33}^- = Ek^2 \pi \dfrac{\partial \mathcal{U}_3^0}{\partial x_3} \text{ on } \omega \times \{0\}, \end{cases} \tag{9.3}$$

• the boundary conditions:

$$\begin{cases} \dfrac{\partial \mathcal{U}_3^0}{\partial x_3} = 0 \text{ on } \omega \times \{L\}, \\ \\ \sigma_{\alpha 3}^- = \sigma_{33}^- = 0 \text{ on } \omega \times \{-l\}, \\ \\ u^- = 0 \text{ on } \partial\omega \times \, ]-l, 0[. \end{cases} \tag{9.4}$$

According to the proof developed in the previous sections, we can state the following convergence result:

**Theorem 9.1.** *Under the assumptions (2.19)÷(2.21) on the applied forces, the sequence $(u^\varepsilon, \sigma^\varepsilon)$ satisfy the following convergences:*

- $\varepsilon \mathcal{T}^\varepsilon(u_\alpha^\varepsilon) \to u_\alpha^0$ *strongly in* $L^2(\omega, H^1(D \times ]0,L[))$, *for* $\alpha = 1, 2$,

$$\bullet\ \mathcal{T}^\varepsilon(u_3^\varepsilon) \to u_3^0 \textit{ strongly in } L^2(\omega, H^1(D\times]0,L[)),$$

*where* $(u_1^0, u_2^0, u_3^0)$ *is the Bernoulli-Navier displacement*

$$u_\alpha^0(x_1,x_2,x_3,X_1,X_2) = \mathcal{U}_\alpha^0(x_1,x_2,x_3), \textit{ for } \alpha = 1,2,$$

$$u_3^0(x_1,x_2,x_3,X_1,X_2) = \mathcal{U}_3^0(x_1,x_2,x_3) - kX_1\frac{\partial \mathcal{U}_1^0}{\partial x_3}(x_1,x_2,x_3) - kX_2\frac{\partial \mathcal{U}_2^0}{\partial x_3}(x_1,x_2,x_3),$$

$\mathcal{U}_1^0$, $\mathcal{U}_2^0$ *and* $\mathcal{U}_3^0$ *being the solution of (9.1), and (9.2)÷(9.4).*

$$\bullet\ \varepsilon\mathcal{U}_\alpha^\varepsilon \to \mathcal{U}_\alpha^0 \textit{ strongly in } L^2\left(\omega, H^1(]0,L[)\right), \textit{ for } \alpha = 1,2,$$

$$\bullet\ \mathcal{U}_3^\varepsilon \to \mathcal{U}_3^0 \textit{ strongly in } L^2\left(\omega, H^1(]0,L[)\right),$$

$$\bullet\ \mathcal{T}^\varepsilon(\gamma_{ij}(u^\varepsilon)) \to X_{ij} \textit{ strongly in } L^2(\Omega^+\times D), \textit{ for } i,j = 1,2,3,$$

*where*

$$X_{11} = X_{22} = \nu\left\{-\frac{\partial \mathcal{U}_3^0}{\partial x_3} + kX_1\frac{\partial^2 \mathcal{U}_1^0}{\partial x_3^2} + kX_2\frac{\partial^2 \mathcal{U}_2^0}{\partial x_3^2}\right\},$$

$$X_{12} = X_{13} = X_{23} = 0,$$

$$X_{33} = \frac{\partial \mathcal{U}_3^0}{\partial x_3} - kX_1\frac{\partial^2 \mathcal{U}_1^0}{\partial x_3^2} - kX_2\frac{\partial^2 \mathcal{U}_2^0}{\partial x_3^2}.$$

$$\bullet\ \mathcal{T}^\varepsilon(\sigma_{ij}^\varepsilon) \to \Sigma_{ij} \textit{ strongly in } L^2(\Omega^+\times D), \textit{ for } i,j = 1,2,3$$

*where*

$$\Sigma_{11} = \Sigma_{22} = \Sigma_{12} = \Sigma_{13} = \Sigma_{23} = 0,$$

$$\Sigma_{33} = E\left(\frac{\partial \mathcal{U}_3^0}{\partial x_3} - kX_1\frac{\partial^2 \mathcal{U}_1^0}{\partial x_3^2} - kX_2\frac{\partial^2 \mathcal{U}_2^0}{\partial x_3^2}\right),$$

*with* $E = \dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$.

$$\bullet\ u_i^\varepsilon \to u_i^- \textit{ strongly in } H^1(\Omega^-), \textit{ for } i = 1,2,3,$$

$$\bullet\ \sigma_{ij}^\varepsilon \to \sigma_{ij}^- \textit{ strongly in } L^2(\Omega^-), \textit{ for } i,j = 1,2,3.$$

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